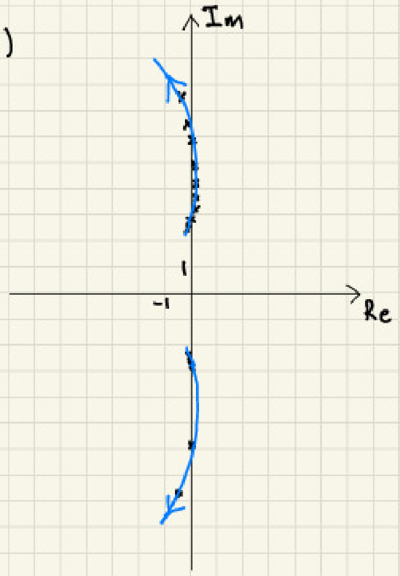
4F1 Solutions 2023

1. (a) Closed-loop stable if and only if $-0.01 < k < 0.25$ or $k > 1.6$. [10%]

(b)

k	roots
$\frac{1}{6}$	$-0.08 \pm j2.4$
$\frac{1}{5}$	$-0.04 \pm j2.6$
$\frac{1}{4}$	$\pm j2.8$
$\frac{1}{3}$	$0.05 \pm j3.2$
$\frac{1}{2}$	$0.1 \pm j3.7$
$\frac{2}{3}$	$0.12 \pm j4.2$
1	$0.11 \pm j4.9$
$\frac{8}{5}$	$\pm j5.9$
2	$-0.1 \pm j6.5$
3	$-0.43 \pm j7.6$

(c)



[25%]

[15%]

(d) Gain margin $= 20\log_{10}(3/1.6) = 5.46$ dB. This is the smallest gain change which makes the system unstable and here corresponds to a gain *reduction*, so the system is *conditionally stable*. This is not satisfactory in practical implementation if it can be avoided. [10%]

(e) Phase lead is required to bring the Nyquist below the negative real axis. A little phase advance is needed between the frequencies of 2.8 and 5.9 rad/sec. (The minimum phase is around -185^o). $\omega_c = 4$ and $\alpha = \sqrt{10}$ is more than sufficient (and is far more than is actually required to avoid conditional stability) and it means a phase margin of at least 50^o would be achieved for any choice of k. [20%]

(f) With K_1 as selected this gives $L(0) = 100k/\alpha$. A phase lag compensator with a low frequency gain of α and a high frequency gain of 1 and with break frequencies about a decade below those of the lead compensator satisfies the specification:

$$K_2(s) = \frac{s + 0.03\alpha}{s + 0.03}$$

[Note for reference only: the actual transfer function in the question was: $G(s) = (s+10)^2/(s+1)^3$.] [20%]

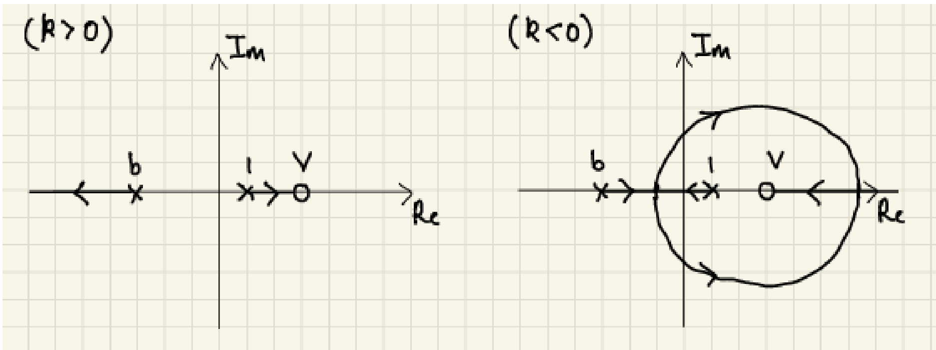
Examiner's comment. The least popular question but with mostly very good attempts. In Parts (b) and (c) a few candidates failed to identify the correct direction of increasing frequency, failing to notice that it can be deduced from the fact that $|G(j\omega)| \to 0$ as $\omega \to \infty$. Few candidates fully explained the issue of conditional stability in Part (d). Parts (e) and (f) were generally well done.

2. (a) $L(s) = G_1(s)K(s) = k/(s-1)$ hence $S(s) = (s-1)/(s+k-1)$ which is stable for $k > 1$ and has $|S(j\omega)| \leq 1$ for all ω providing $k \geq 2$. [15%]

   (b) (i) Breakaway points are the roots of

$$\begin{aligned} 0 &= (s-V)(2s+b-1) - (s^2 + (b-1)s - b) \\ &= s^2 - 2Vs + b - Vb + V \end{aligned}$$

which has a left half plane root if and only if $b - Vb + V < 0$. (Since sum of roots is positive the produce of roots needs to be negative. Or note that we need "4ac" to be negative to get a left half plane root from the formula for the roots of a quadratic.)



[20%]

(ii) By applying first a compensator of the form $(s+1)/(s+b)$ the task of stabilising $G_2(s)$ becomes the task of stabilising $L_0(s)$ which can be achieved with a constant gain if $b > \frac{V}{V-1}$, so we need to choose first a b satisfying this and then proceed to find a stabilising k.
Take $V = 2$ and $b = 5$. Closed loop poles of $L_0(s)$ with gain k are roots of

$$\begin{aligned} 0 &= (s+5)(s-1) + k(s-2) \\ &= s^2 + (4+k)s + (-2k-5) \end{aligned}$$

which has stable roots with $k = -3$. Hence writing

$$-3L_0(s) = -3\frac{s-2}{(s-1)(s+5)} = \frac{s-2}{s^2-1}\,\frac{-3(s+1)}{s+5}$$

shows that $K(s) = -3(s+1)/(s+5)$ is a stabilising compensator for $G_2(s)$. [15%]

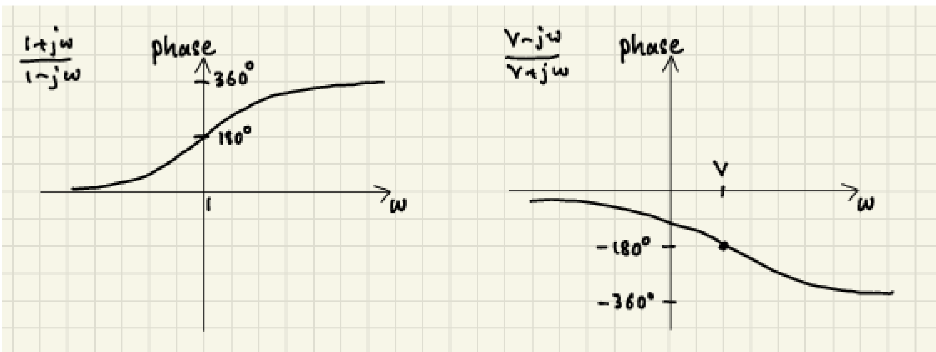
(iii) In the root-locus of $G_2(s)$ the number of roots to the right of the vertical line through $s = V$ can only change by an even number as pairs of roots cross this line. Since there is one root at $s = 1$ to the right of $s = V$ the system can't be stabilised unless the compensator provides an odd number of poles to the right of $s = V$. [15%]

(iv)

$$\begin{aligned} G_2(s) &= G_m(s)B_p(s)B_z(s) \\ &= \frac{V+s}{(1+s)^2}\,\frac{1+s}{1-s}\,\frac{V-s}{V+s} \end{aligned}$$

[10%]

(v) The rising phase characteristic in $B_p(s)$ is needed to provide the anti-clockwise encirclement to satisfy the Nyquist stability criterion. If V is much greater than 1 the phase lag from $B_z(s)$ doesn't enter until high frequency making the stabilisation task easier. If $V = 1$ stabilisation is theoretically impossible, and if $0 < V < 1$ the task is very difficult, as already seen.



[15%]

(vi) Wear a helmet! Theory says that stabilisation gets easier for larger V - but the rider needs to be very brave to get above the low speeds when stabilisation is very tricky or impossible. Think twice before attempting this! [10%]

Examiner's comment. Part (a) was rarely completely correct with few candidates stating the condition on k necessary for stabilisation. Parts (b)(i)-(ii) were mostly well done whereas in Part (b)(iii) candidates were able to explain things roughly though no candidate gave a full justification. In Part (iv) a surprising number of candidates didn't normalise the all-pass factors at $s = 0$ despite the explicit instruction to do so. Rather few were able to draw the same conclusions on difficulty of control from the phase characteristic in Part (b)(v) as had been shown in previous question parts by root-locus considerations. In Part (b)(vi) one or two did correctly recommend a helmet should be worn!

3. (a) Since one of the branches will be asymptotic to the zero then $z > 0$ is necessary. The asymptote centre is equal to $(3-p+z)/2$ hence $p > z + 3$ is also necessary. With $k > 0$ the asymptotes are parallel to the imaginary axis (from the real axis rule since the total number of poles and zeros is even) so $p - 3 > z > 0$ is sufficient for stability for large k. [20%]

(b) Closed-loop poles are roots of

$$\begin{aligned} 0 &= (s+1)\,(s(s-4)(s+10)+k(s+a)) \\ &= (s+1)\left(s^3+6s^2+(k-40)s+ak\right) \end{aligned}$$

so, from the Routh-Hurwitz criterion, a necessary and sufficient condition for stability is that: $k > 40$, $a > 0$, $6(k-40) > ak$. These reduce to:

$$0 < a < 6, \quad k > \frac{240}{6-a}$$

[15%]

(c) (i)

$$E(s) = \frac{1}{1+G(s)K(s)}\,\frac{1}{s} = \frac{(s-4)(s+10)}{s(s-4)(s+10)+k(s+a)}$$

so $E(0) = -40/(ka) < 0$. [10%]

(ii) Putting $s = 0$ in the definition of the Laplace transform of $e(t)$ gives:

$$E(0) = \int_0^\infty e(t)dt$$

and the result follows from the previous part. [10%]

(iii) It follows that $e(t)$ must be negative (i.e. $y(t)$ must exceed $r(t)$) at some positive times t. [10%]

(d) (i) $2/((s+1)(s+2))$ has second-order roll-off at high frequency (which is not less than that of $G(s)$) and there are no RHP zeros in $G(s)$ hence this transfer function is achievable in a two-degree-of-freedom design. From:

$$\frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

the step response is equal to: $1 - 2e^{-t} + e^{-2t} = (1-e^{-t})^2 \leq 1$ so there is no overshoot. [15%]

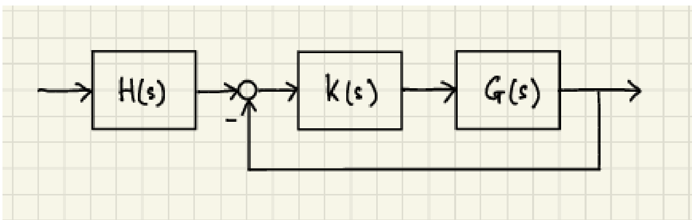
(ii) Note that

$$\frac{GK}{1+GK} = \frac{k(s+a)}{s^3 + 6s^2 + (k-40)s + ak}$$

so we can choose

$$H(s) = \frac{2(s^3 + 6s^2 + (k-40)s + ak)}{k(s+a)(s+1)(s+2)}$$

in the block diagram below, with a and k selected as in Part (b).



[20%]

Examiner's comment. The most popular question. In Part (a) most candidates got the condition on z and p for the asymptote centre to be in the LHP but many forgot the condition on z and didn't remark on the asymptote orientation. Parts (b) and (c) were mostly correctly done. In Part (d)(i) many overlooked the need to check "suitability", namely that the specified transfer function did not have any overshoot. In Part (d)(ii) there were a number of attempts that did not seek a pre-filter but tried to find a $K(s)$ to do the job and then ending up with a pole-zero cancellation at $s = 4$ without noticing that this is not allowed.

M.C. Smith, 4 May 2023