Version MCS/3

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 26 April 2023 2 to 3.40

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F1 Formulae sheet (3 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 Figure 1 shows the locus of $G(j\omega)$ for positive ω in various portions of the G-plane for a stable transfer function $G(s)$ satisfying $|G(j\omega)| \to 0$ as $\omega \to \infty$. Figures 1(b)-(d) additionally show the image of a *rectangular* grid in the s-plane. For $s = \sigma + j\omega$ the images for $\sigma = -0.6, -0.5, \ldots, 0.6$ (at intervals of 0.1) and for $\omega = 2.5, 3.0, \ldots, 9.0$ (at intervals of 0.5) are shown as (unlabelled) dashed lines. $G(j\omega)$ intersects the real axis at exactly the points: $-4, -0.625, 100$.

(a) Use the Nyquist stability criterion to determine the range of k (both positive and negative) for which the closed-loop system is stable with constant gain negative feedback k. [10%]

(b) Estimate the closed-loop pole locations near to the imaginary axis for constant gain negative feedback with values $k = \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{8}{5}, 2, 3$. [25%]

(c) Use your answer to Part (b) to sketch a portion of the root-locus diagram for $G(s)$. [15%]

(d) What is the gain margin in decibels if negative feedback of $k = 3$ is applied? Explain the disadvantage of this choice of feedback gain. [10%]

(e) Suggest values of α and ω_c so that $G(s)$ is stabilised (in the negative feedback convention) by

$$K_1(s) = k\,\frac{\alpha s + \omega_c}{s + \omega_c \alpha}$$

for all $k > 0$. [20%]

(f) The compensator $K_1(s)$ of Part (e) is selected with a proposed value of $k = 3$. Suggest a further stage of compensation which would allow $L(0) = 300$, where $L(s)$ is the return ratio of the feedback loop, without increasing the high frequency loop gain and while maintaining the property that closed loop stability is achieved for any $k > 0$. [20%]

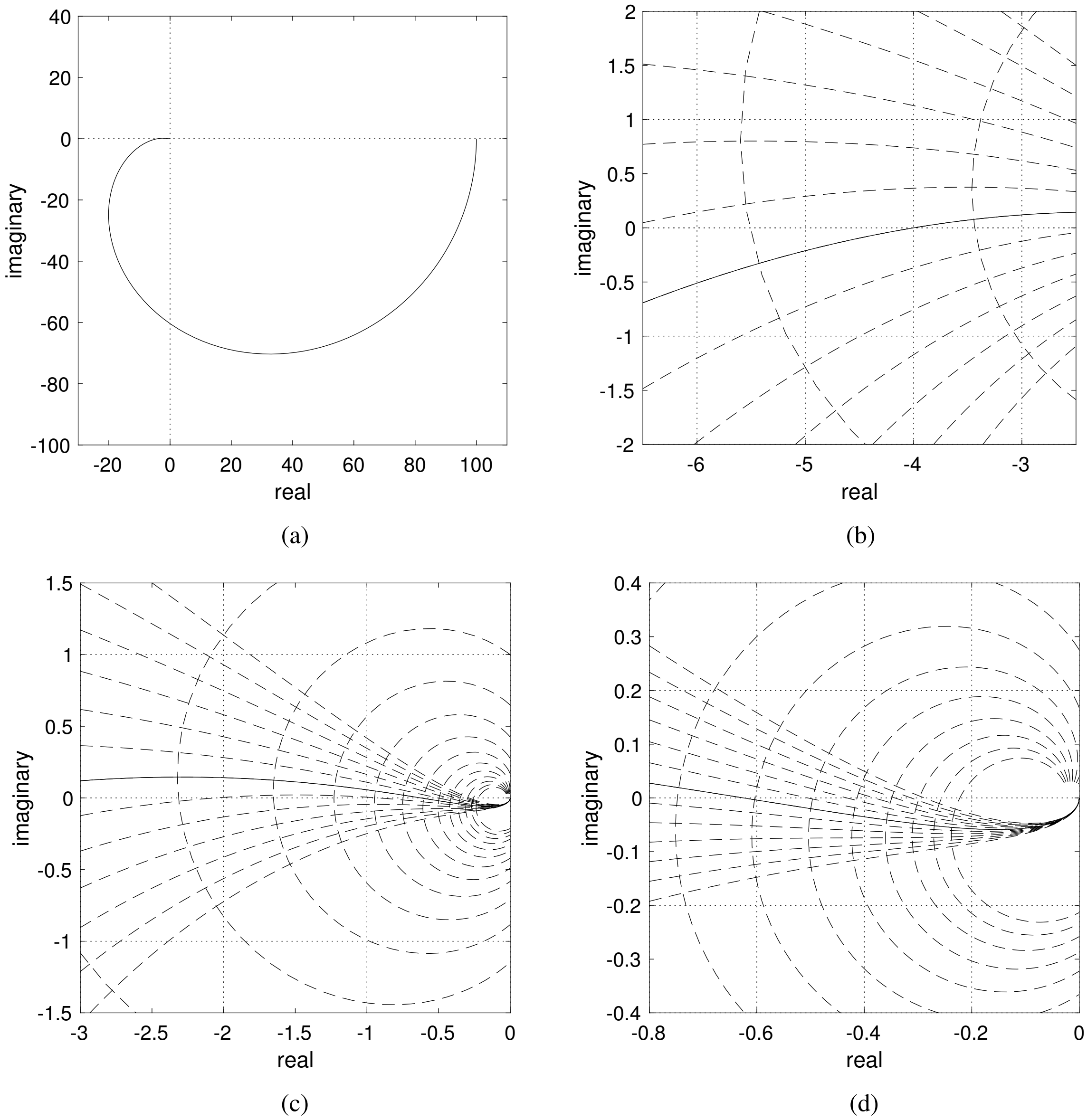

40
20
0
-20
-40
-60
-80
-100
imaginary
-20 0 20 40 60 80 100
real
(a)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
imaginary
-6 -5 -4 -3
real
(b)
1.5
1
0.5
0
-0.5
-1
-1.5
imaginary
-3 -2.5 -2 -1.5 -1 -0.5 0
real
(c)
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
imaginary
-0.8 -0.6 -0.4 -0.2 0
real
(d)

Fig. 1

2 (a) A simplified (and normalised) linear model for a bicycle has a transfer function from steering angle input to tilt angle equal to:

$$G_1(s) = \frac{s+V}{s^2-1}$$

where V is the forward velocity. By considering a compensator of the form

$$K(s) = k\frac{s+1}{s+V}$$

for some constant k, show that the bicycle may be controlled so that $|S(j\omega)| \leq 1$ for all ω, where $S(s)$ is the sensitivity function. [15%]

(b) It is desired to study the control of a rear-wheel steered bicycle using the simplified model of Part (a) but with the sign of the velocity reversed. Accordingly the transfer function is modified to:

$$G_2(s) = \frac{s-V}{s^2-1}$$

where V is the forward velocity of this bicycle.

(i) Consider the transfer function

$$L_0(s) = \frac{s-V}{(s-1)(s+b)}$$

for $V > 1$ and $b > 0$. Show that there is a breakaway point of the root-locus for $L_0(s)$ in the left half-plane if and only if $b - Vb + V < 0$, or equivalently

$$b > \frac{V}{V-1},$$

and sketch the root-locus for positive and negative gain k (with the usual negative feedback configuration) when this condition holds. [20%]

(ii) Deduce that there is a stable, stabilising compensator for $G_2(s)$ when $V > 1$ and find such a compensator when $V = 2$. [15%]

(iii) Using root-locus considerations, or otherwise, show that $G_2(s)$ cannot be stabilised with a *stable* compensator $K(s)$ when $0 < V < 1$. [15%]

(iv) Express the transfer-function in the form

$$G_2(s) = G_m(s)B_p(s)B_z(s)$$

where $B_p(s)$ is a pole-type all-pass function with $B_p(0) = 1$, $B_z(s)$ is a zero-type all-pass function with $B_z(0) = 1$, and $G_m(s)$ has no poles or zeros with $\mathrm{Re}(s) > 0$. [10%]

(v) With reference to the phase of $B_z(j\omega)$ for differing values of V, and with the aid of suitable Bode plots, explain how the difficulty of control of $G_2(s)$ relates to the value of V. [15%]

(vi) What advice would you give to a person who is attempting to learn to ride a rear-wheel steered bicycle? [10%]

3 An electromagnetic levitation system is required to achieve accurate position control in a confined space. The transfer function of the system from actuator input to position is given by

$$G(s) = \frac{1}{(s-4)(s+1)}.$$

(a) For a compensator of the form $K_0(s) = k(s+z)/(s+p)$, by considering the asymptote centre and asymptotic behaviour of the root-locus, or otherwise, find conditions on z and p so that $G(s)$ is stabilised for large k in a negative feedback configuration. [20%]

(b) A lead compensator in the form $K_1(s) = (s+1)/(s+10)$ is selected together with a proportional-plus-integral controller of the form $K_2(s) = k(s+a)/s$ for some positive constants k and a, to form an overall compensator in the form $K(s) = K_1(s)K_2(s)$. Use the Routh-Hurwitz stability criterion to find necessary and sufficient conditions on k and a for closed-loop stability in the standard negative feedback configuration. [Hint: the root at $s = -1$ can be factored out.] [15%]

(c) Let $e(t) = r(t) - y(t)$ be the error signal between reference input $r(t)$ and position output $y(t)$. Suppose a stabilising pre-compensator $K(s)$ of the form suggested in Part (b) is selected.

(i) For $r(t)$ equal to a step input show that $E(0) < 0$ where $E(s)$ is the Laplace transform of $e(t)$. [10%]

(ii) Deduce that

$$\int_0^\infty e(t)dt < 0.$$

[10%]

(iii) Explain why this control scheme gives overshoot in $y(t)$ for a step input $r(t)$. [10%]

(d) A two-degree-of-freedom control scheme is desired which has no overshoot in the step response.

(i) Show that

$$\frac{2}{(s+1)(s+2)}$$

is a suitable achievable closed-loop transfer-function from $r(t)$ to $y(t)$. [15%]

(ii) Design a control scheme to achieve this closed-loop transfer function. [20%]

END OF PAPER

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

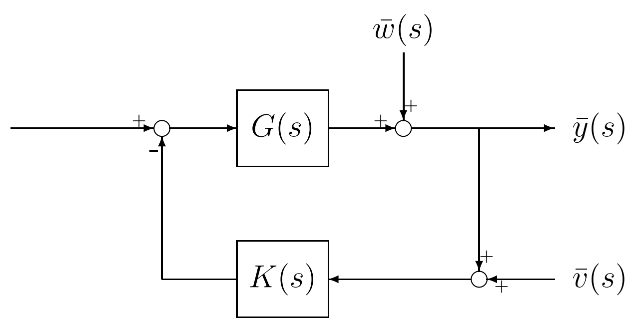
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2\arctan\alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\, dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

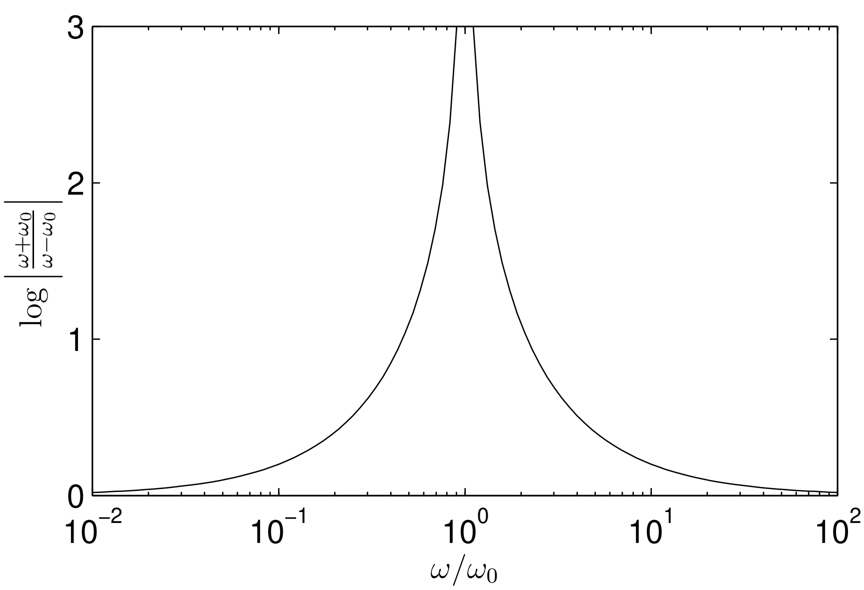


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v)|}{dv}\right|_{v=0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^v\big)\right| dv}$$

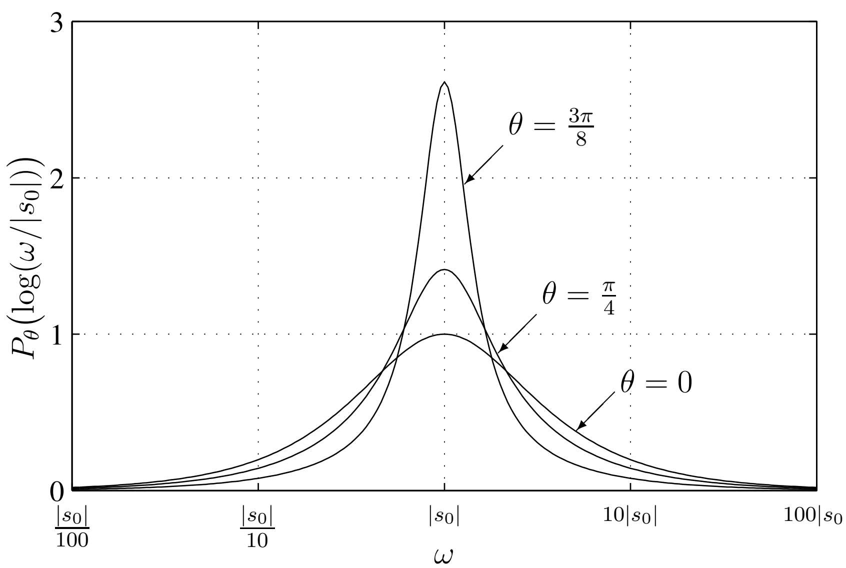
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith
November 2022