EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 24 April 2024   2 to 3.40

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F1 Formulae sheet (3 pages)
Supplementary pages: two extra copies of Fig. 2 (Question 2)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) The plant with transfer function

$$G(s) = \frac{2s}{s^2 + 1}$$

is to be controlled with constant gain feedback k in the standard negative feedback configuration.

(i) Sketch the Nyquist contour and the complete Nyquist diagram for $G(s)$. Use the Nyquist stability criterion to determine the number of closed-loop poles in the right half-plane for all values of k both positive and negative. [20%]

(ii) Find all frequencies for which $|G(j\omega)| = 1$. Hence find the smallest time delay that can de-stabilise the system when $k = 1$. [20%]

(b) An automatic stabilisation system for the attitude control of a vertical takeoff (VTOL) aircraft is shown in Fig. 1 where $\theta(t)$ is the attitude and $r(t)$ is the reference input. The vehicle has transfer function

$$G_0(s) = \frac{1}{s^2 + 1}$$

and the feedback filter is a rate gyro with $F_0(s) = s$. The controller and actuator dynamics are combined in the pre-compensator block $K(s)$ which is assumed to have the form

$$K(s) = \frac{k(s+5)}{s+a}$$

where $|k - 2| \leq \delta$ and $3 \leq a \leq 5$.

(i) Assuming the Small Gain Theorem derive a necessary and sufficient condition for the feedback system of Fig. 1 to be internally stable for all stable $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq h(\omega)$ for the multiplicative uncertainty model $K(s) = K_0(s)(1 + \Delta(s))$. [15%]

(ii) Write $K_0(s) = 2$ and show that all assumed $K(s)$ are contained in the uncertainty model with

$$h(\omega) = \left| \frac{\delta j\omega + (4 + 5\delta)}{2(j\omega + 3)} \right| .$$

[15%]

(iii) Verify that the condition you derived in Part (b)(i) is satisfed when $\delta = 0$. [15%]

(iv) Comment on the sensitivity of the closed-loop transfer function relating attitidue $\theta(t)$ to reference input $r(t)$ at $s = 0$. Is there any way that the sensitivity could be reduced at $s = 0$? [15%]

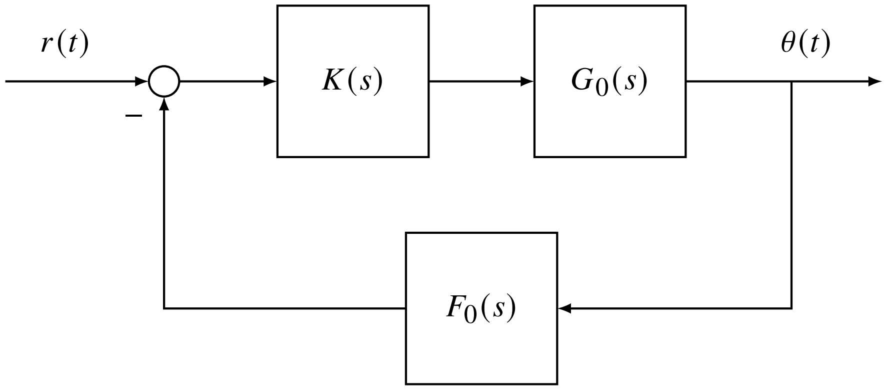

Fig. 1

2 Figure 2 is the Bode diagram of a system with transfer function $G(s)$ for which a compensator $K(s)$ is to be designed. It is known that the system has no poles satisfying $\mathrm{Re}(s) > 0$.

(a) (i) Sketch on a copy of Fig. 2 the expected phase of $G(j\omega)$ if $G(s)$ were stable and minimum phase. [15%]

(ii) Determine the form of the all-pass part of the transfer function and estimate any associated parameters. [10%]

(iii) Comment briefly on any limitations that may be experienced in the design of $K(s)$. [10%]

(b) Suppose it is desired to achieve the following specifications for the return ratio $L(s) = G(s)K(s)$:

A: Velocity error constant $K_v \geq 10$ where $K_v = \lim_{s \to 0} (sL(s))$;

B: Gain cross-over frequency ω_c (i.e. $|L(j\omega_c)| = 1$) satisfying $\omega_c \geq 1 \text{ rad s}^{-1}$;

C: Phase margin of at least 45°;

D: $|L(j\omega)| \leq 0.1$ for $\omega \geq 20 \text{ rad s}^{-1}$.

(i) By considering the cases of a lead and lag compensator separately, or otherwise, explain why it is not possible to achieve the specifications using a compensator with one pole and one zero. [25%]

(ii) Design a compensator $K(s)$ to satisfy specifications A–D. Hint: you may find it helpful to first design a phase lead compensator to satisfy specifications C–D with a gain cross-over frequency $\omega_c = 1 \text{ rad s}^{-1}$. Sketch the Bode diagram of $K(s)$ and $G(s)K(s)$ for your design on a copy of Fig. 2. [40%]

Two copies of Fig. 2 are provided on separate sheets. These should be handed in with your answers.

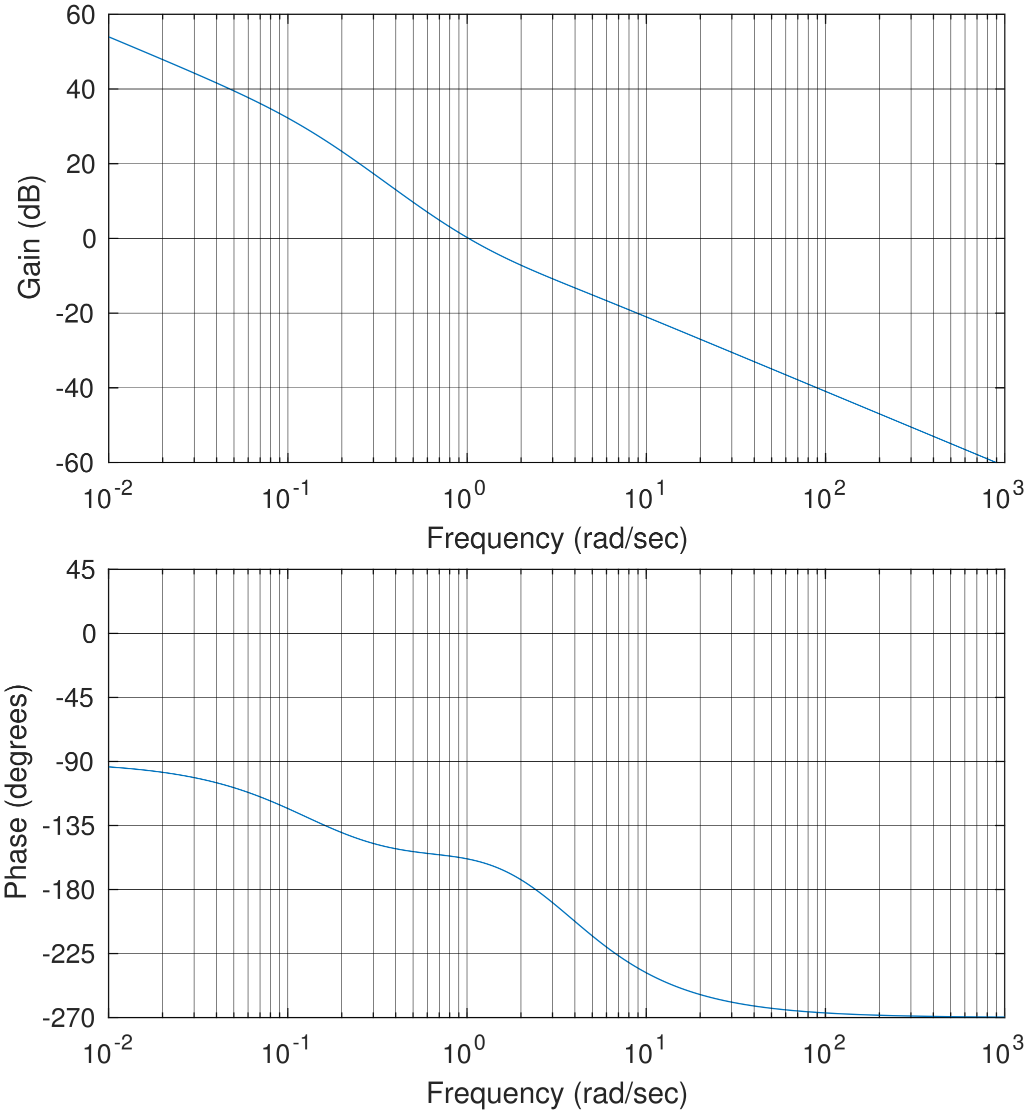

Fig. 2

(TURN OVER

3 A control system is to be designed for a flexible mechanical manipulator whose transfer function from actuator input to measured output takes the form

$$G(s) = \frac{s^2 + 0.1s + 1}{(s^2 + 0.1s + 4)(s - 1)}.$$

(a) Let $T(s)$ denote the Complementary Sensitivity function of an internally stable feedback system containing the plant $G(s)$ and a controller $K(s)$ in the standard negative feedback configuration. It may be assumed that $K(s)$ is minimum phase.

(i) Explain why:

A. $T(1) = 1$; [5%]

B. $\log(T(s))$ is analytic in $\mathrm{Re}(s) > 0$. [10%]

(ii) Show that

$$\int_0^\infty \frac{1}{1 + \omega^2} \ln |T(j\omega)| d\omega = 0.$$

[10%]

(iii) Discuss the consequences of the formula in Part (a)(ii) for the effect of measurement noise at the plant output. [10%]

(iv) Let $L(s) = G(s)K(s)$ denote the return ratio. It is proposed that $K(s)$ is chosen so that

$$L(s) = \frac{k}{s - 1}$$

for suitable k. Explain the disadvantages of this approach. [20%]

(b) (i) Sketch the root-locus diagram of $G(s)$ for gain $k > 0$ in the standard negative feedback configuration. You may use the fact that there are breakaway points located at: -0.65 and -1.86. What value of k places the closed-loop poles furthest to the left in the complex plane? [25%]

(ii) Keeping k as an unknown but stabilising feedback gain design a two-degree-of-freedom control scheme to achieve the closed-loop transfer function

$$\frac{1}{s + 1}$$

from reference input to plant output. [20%]

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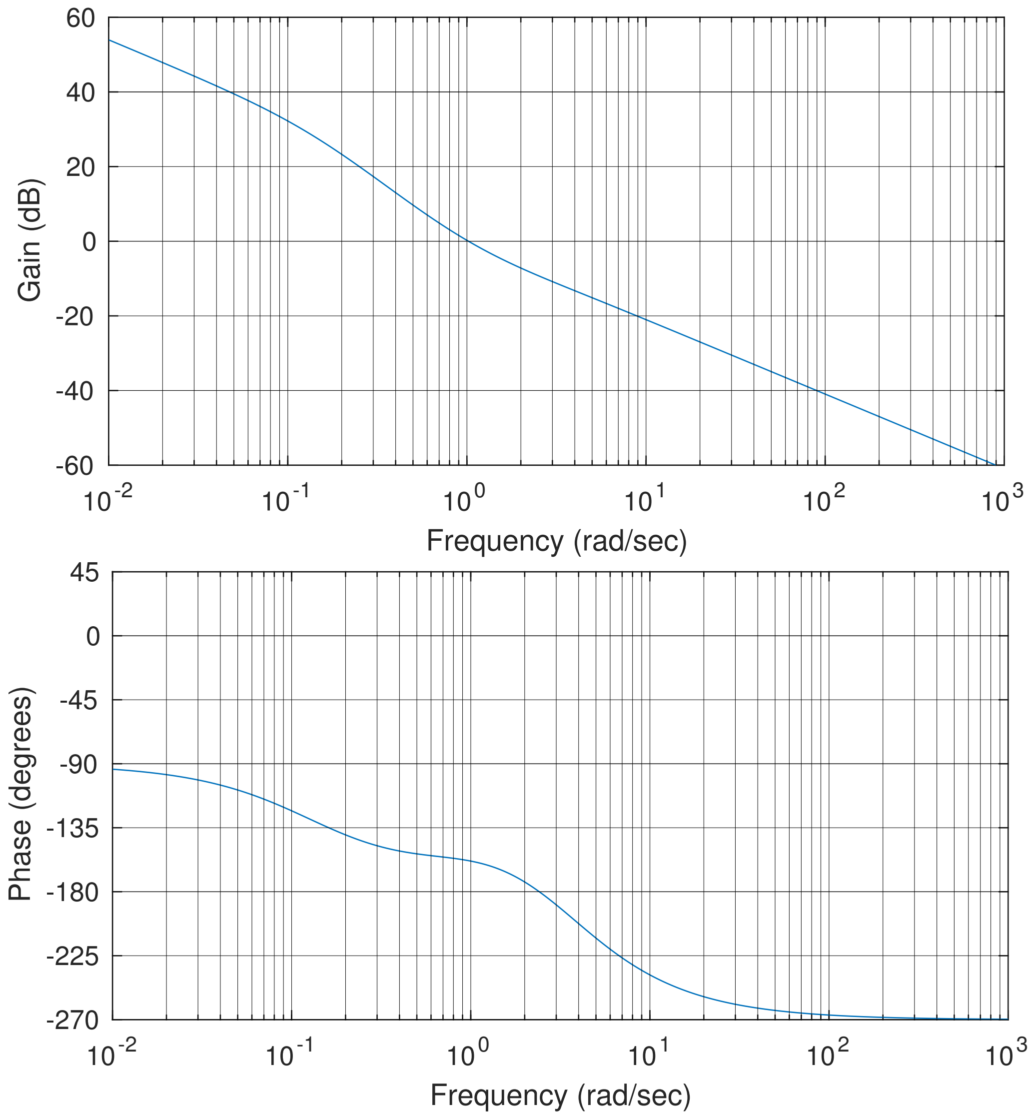

Extra copy of Fig. 2: Bode diagram for Question 2.

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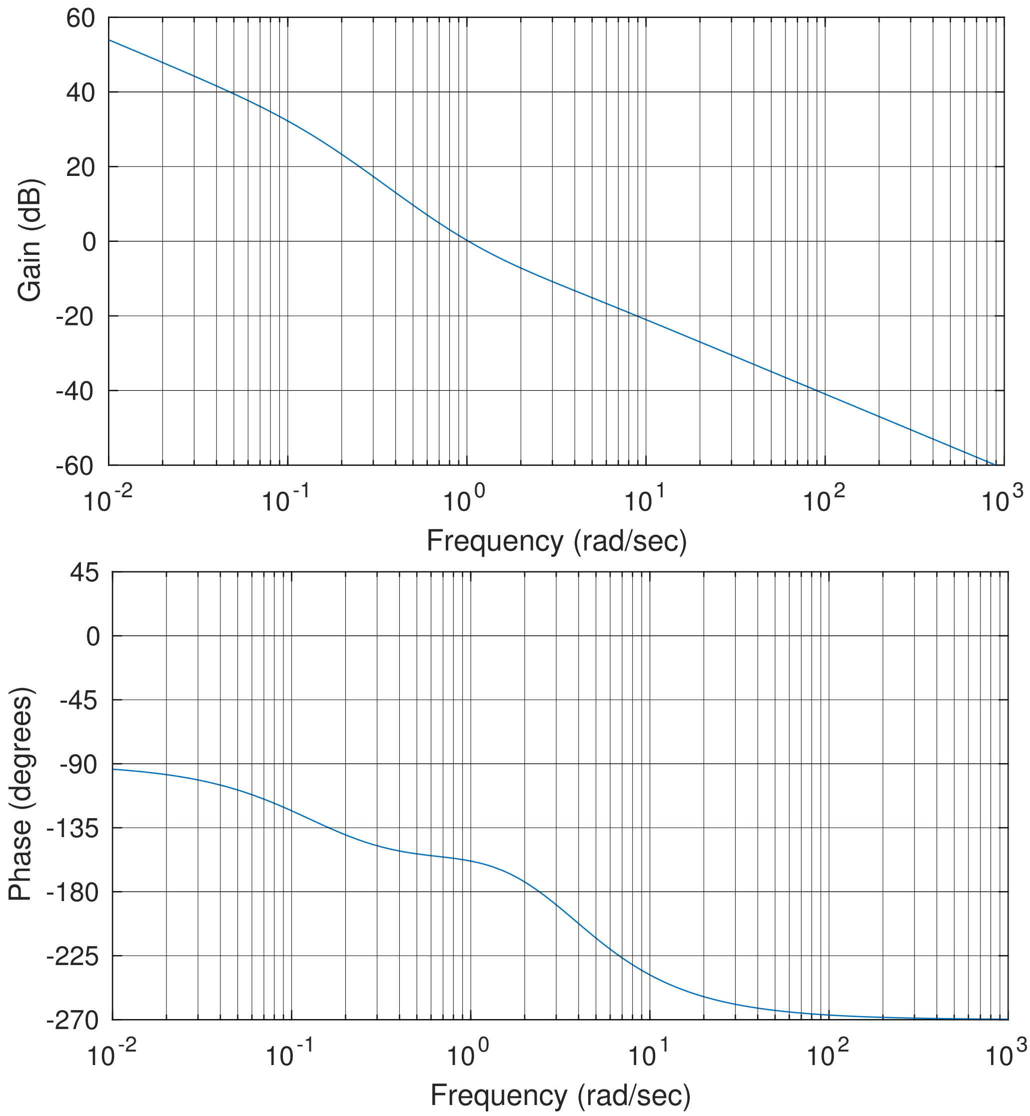

Extra copy of Fig. 2: Bode diagram for Question 2.

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Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

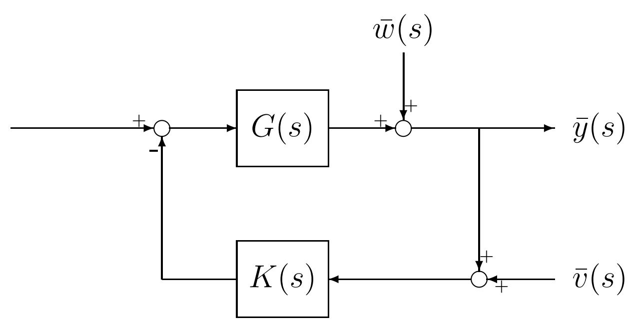
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\, dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

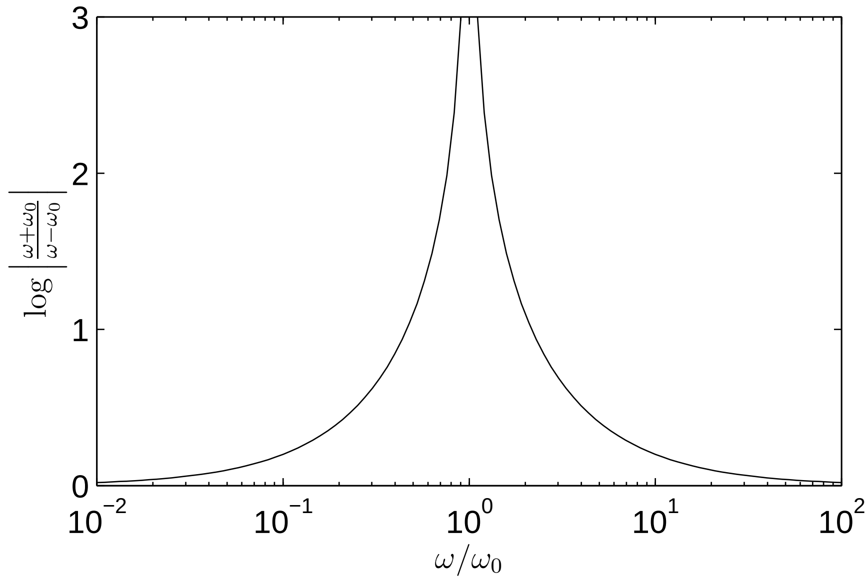


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2} \left.\frac{d\log|L(j\omega_0 e^v)|}{dv}\right|_{v=0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^v\big)\right| dv}$$

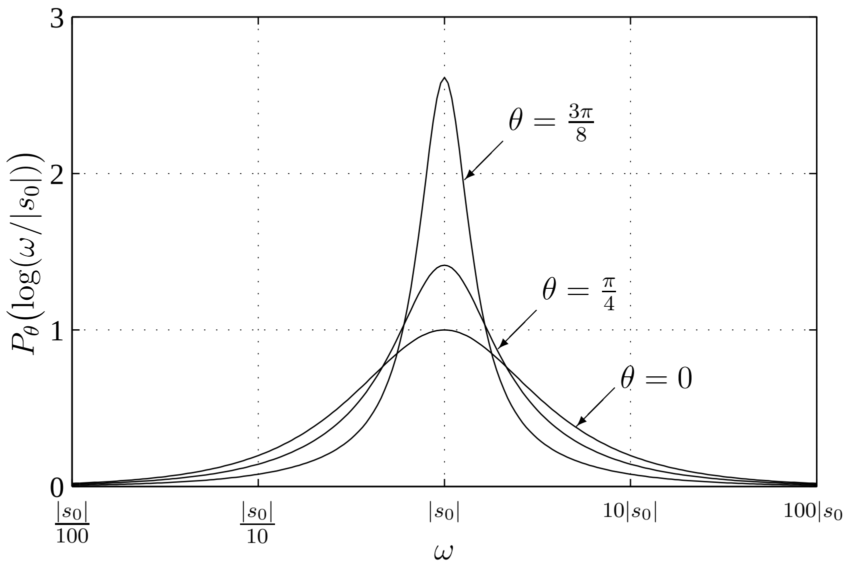
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

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November 2023