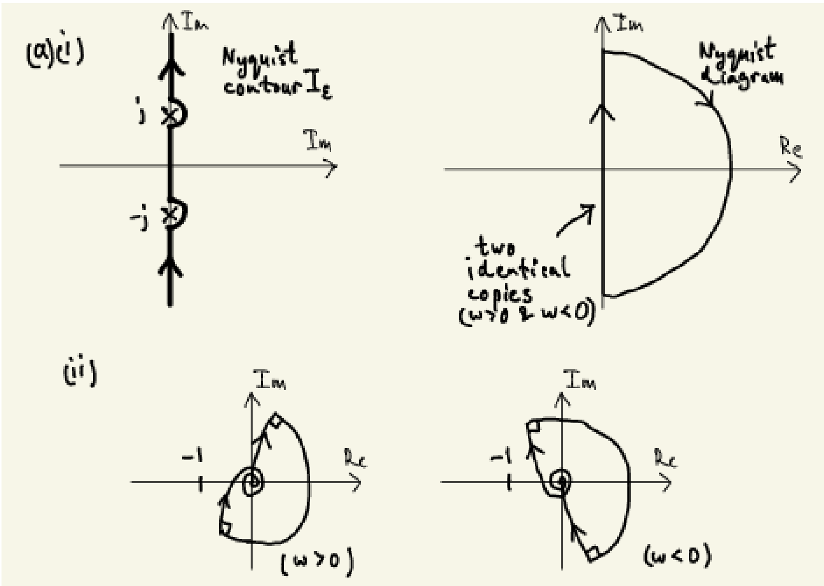
4F1 Solutions 2024

1. (a) (i) The positive and negative frequency portion of the Nyquist contour I_ϵ map to the identical s-plane contour. There are no (two clockwise) encirclements of $-1/k$ for $k > 0$ (resp. $k < 0$). Hence there are no (resp. two) closed-loop poles in the RHP for $k > 0$ (resp. $k < 0$). [20%]



(ii) $|G(j\omega)| = 1$ if and only if $2|\omega| = |1 - \omega^2|$, i.e.

$$\omega^2 + 2\omega - 1 = 0 \text{ or } \omega^2 - 2\omega - 1 = 0$$

with positive solutions $\omega_1 = -1 + \sqrt{2}$ and $\omega_2 = 1 + \sqrt{2}$. From the figure it is the largest solution that causes instability when a time delay is introduced. Smallest destabilising time delay T satisfies $\omega_2 T = \pi/2$ which gives $T = 0.65$ seconds. [20%]

(b) (i) Let $L_0 = K_0 G_0 F_0$ and $T = L_0/(1 + L_0)$. Then standard bookwork shows that a necessary and sufficient condition for the feedback system to be internally stable for all stable $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq h(\omega)$ is that

$$h(\omega) < |T(j\omega)|^{-1}$$

for all ω. [15%]

(ii) Solving for $\Delta(s)$ gives

$$\Delta = \frac{(k-2)s + 5k - 2a}{2(s+a)}$$

which after taking the magnitude is maximised with $a = 3$, i.e.

$$|\Delta(j\omega)|^2 \leq \frac{\delta^2\omega^2 + (4+5\delta)^2}{4(\omega^2+9)}$$

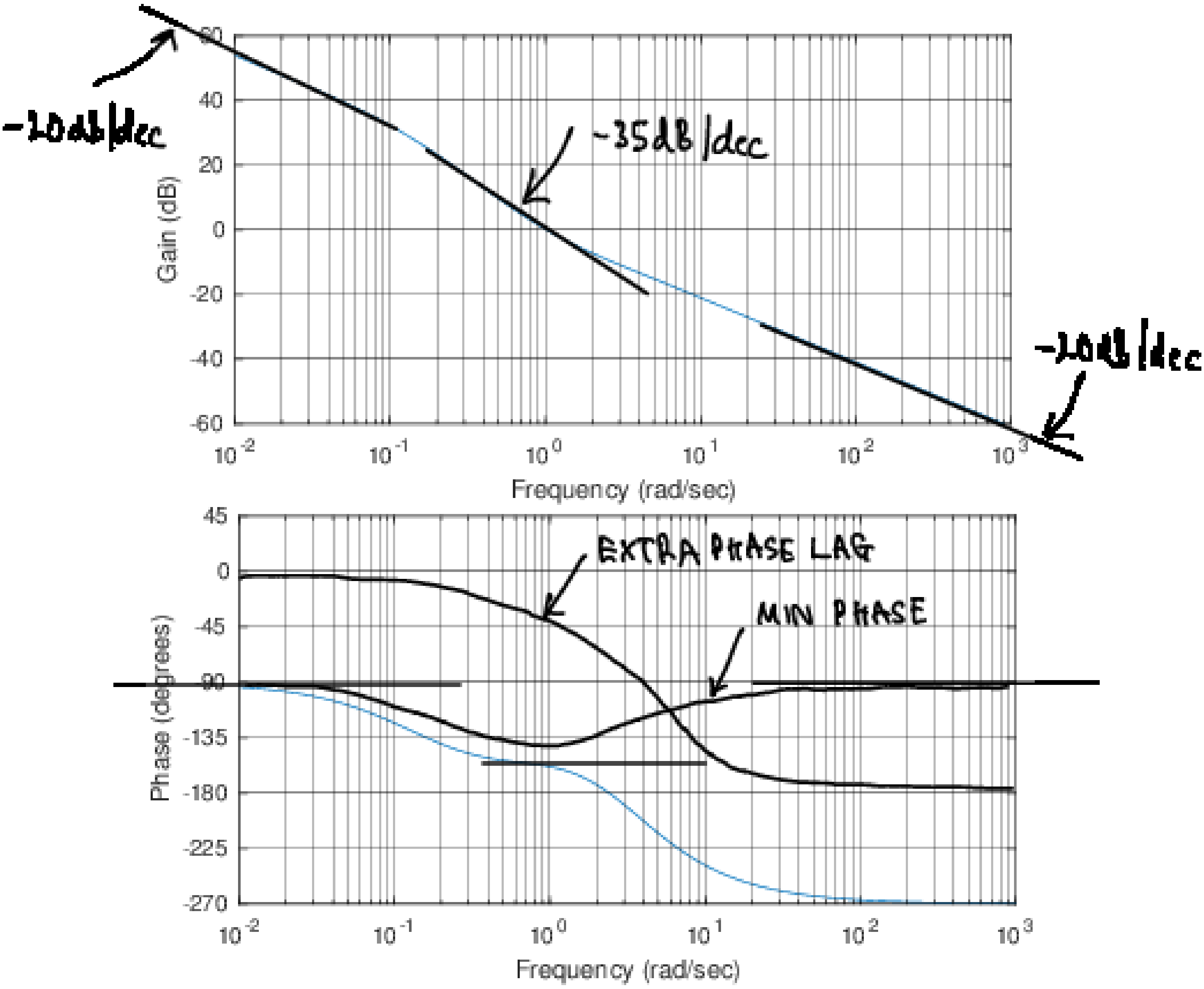
from which the result follows. [15%]

(iii) $L_0(s)$ is equal to $G(s)$ from Part (a) and hence $T(s) = 2s/(s+1)^2$. Note that $|T(j\omega)| = 2|\omega|/(1+\omega^2)$ hence $|T(j\omega)|^{-1} \geq 1$ for all ω. Also $h(\omega) < 1$ for all ω when $\delta = 0$. Hence, clearly, the required inequality is satisfied for all ω. [15%]

(iv) The sensitivity function $S(s) = 1/(1 + L_0(s))$ (it doesn't matter that the variations are in K rather than G). Hence $S(0) = 1$ so there is no sensitivity reduction. This persists as long as $L_0(0) = 0$ which comes from the rate gyro. Compensation can't change this. The only solution would be to change this block, for example by adding a proportional term to the rate gyro feedback, but this means finding a way to measure the attitude. [15%]

Examiner's comment. Sketching the Nyquist diagram for this simple transfer-function caused difficulties for many candidates with few able to combine the large semi-circles from the indentations around the imaginary axis poles with the imaginary axis sections. Few candidates correctly selected the relevant frequency in 1(a)(ii) to calculate the smallest time delay. Part (b)(i) and (iv) were generally well done but (ii) and (iii) had a lot of imprecise manipulations.

2. (a) (i) Annotating by hand with straight-line approximations: [15%]

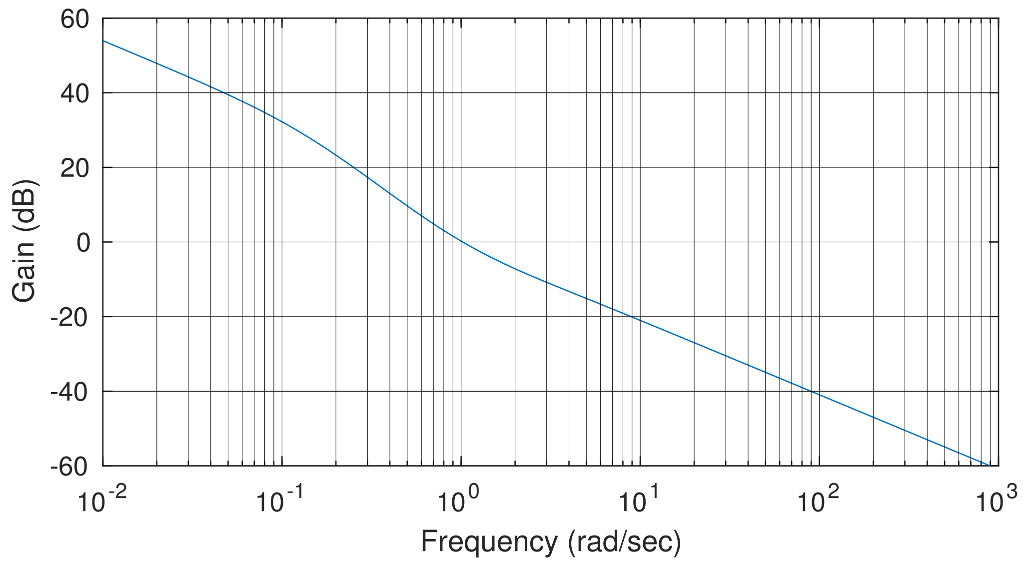


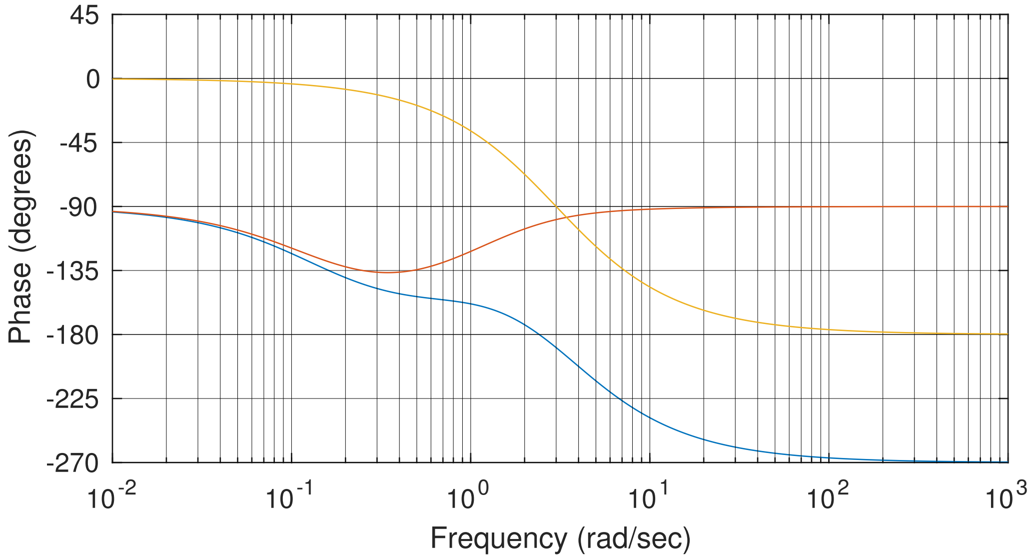
[10%]

(ii) This suggests an all-pass factor of the form

$$\frac{a-s}{a+s}$$

with $a = 4$. True value was $a = 3$. See accurate computer-generated plot overleaf.

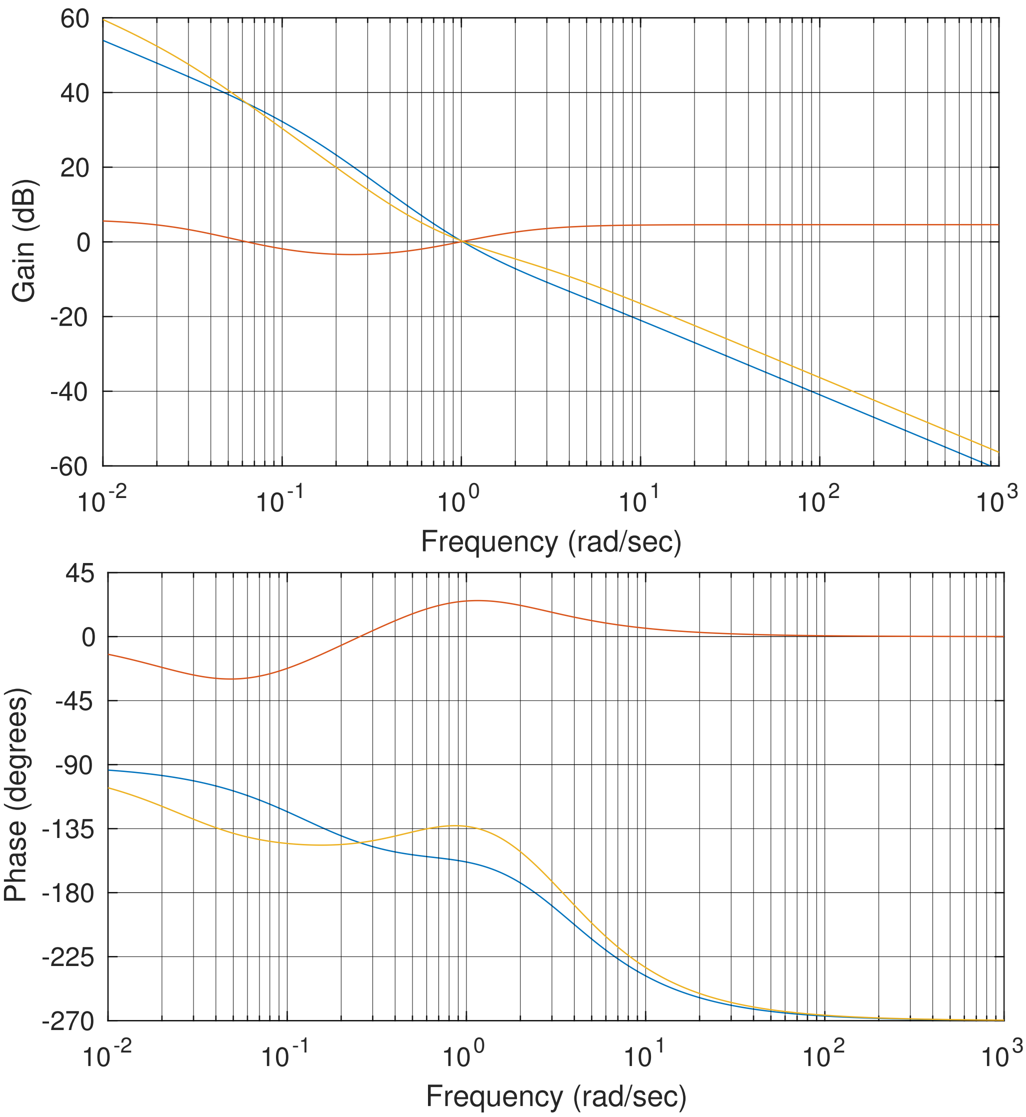




(iii) Difficult to achieve a gain crossover frequency much greater than 3 rad/sec. [10%]

(b) (i) Since the phase of $G(j\omega)$ is lower than -135^o for $\omega \geq 1$ a phase lag compensator alone can never satisfy spec C. A constant gain compensator with gain around 2 can meet both specs A and D (just!). A phase lead compensator, which is needed to achieve spec C, always has higher gain at high frequency compared to low frequency. Once this is introduced specs A and D can't be achieved simultaneously. [20%]

(ii) A phase lead compensator $K_1(s) = \alpha \frac{s+1/\alpha}{s+\alpha}$ with $\alpha = 1.7$ gives 29^o of peak phase advance at $w = 1$ allowing spec C to be satisfied without violating spec D. A phase lag compensator with break frequencies place at low enough frequency so as not to compromise spec C can meet spec A, e.g. $K_2(s) = \frac{s+0.1}{s+0.1/(2\alpha)}$ to give a final compensator $K(s) = K_1(s)K_2(s)$ as shown.



[40%]

Examiner's comment. This question was generally very well done by most candidates who showed an excellent grasp of the Bode gain-phase relationships and the purpose and use of lead and lag compensators. Most candidates produced excellent final designs for Part (b)(ii).

3. (a) (i) A. Write $G(s) = G_1(s)/(s-1)$ where

$$G_1(s) = \frac{s^2 + 0.1s + 1}{s^2 + 0.1s + 4}.$$

Then $T(s) = G_1(s)K(s)/((s - 1 + G_1(s)K(s))$ and $K(1) \neq 0$ because of internal stability, also $G_1(1) \neq 0$ so $T(1) = 1$; [5%]

B. $T(s)$ has no poles in $\mathrm{Re}(s) > 0$ because of internal stability and no zeros in $\mathrm{Re}(s) > 0$ because $K(s)$ has no zeros in $\mathrm{Re}(s) > 0$. Hence $\log(T(s))$ is analytic in $\mathrm{Re}(s) > 0$. [10%]

(ii) From Poisson's formula (data sheet)

$$0 = \log(T(1)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + \omega^2} \log(T(j\omega)) d\omega.$$

Taking real parts of the right hand side and using symmetry gives the result. [10%]

(iii) The formula shows that $|T(j\omega)|$ cannot be less than one at all frequencies. The *waterbed effect* comes into play, i.e. if a design tries to push down $|T(j\omega)|$ in some frequency range it will force it to be larger in others. [10%]

(iii) Such a design means choosing $K(s) = kG_1(s)^{-1}$ which means that the lightly damped poles and zeros in the plant are cancelled by the controller. Some disadvantages are:

A. Input disturbances could excite the lightly damped poles in the plant with the response seen at the output.

B. Because a lightly damped pair of poles is introduced into the controller, sensor noise could be amplified at those frequencies at the plant input.

C. Opportunity is lost to increase the damping of the lightly damped modes.

D. The exact cancellation approach could be susceptible to model errors. [20%]

(b) (i) Real-axis rule together with knowledge of breakaway points suggests the general form of the root-locus. Further analysis could be carried out to check that there is no incursion of the lightly damped poles into the RHP. Angle condition could be used to find angle of departure of lightly damped poles. Closed-loop poles are furthest to the left when

$$k = -1/G(-0.65) = 5.3 \text{ approx}$$

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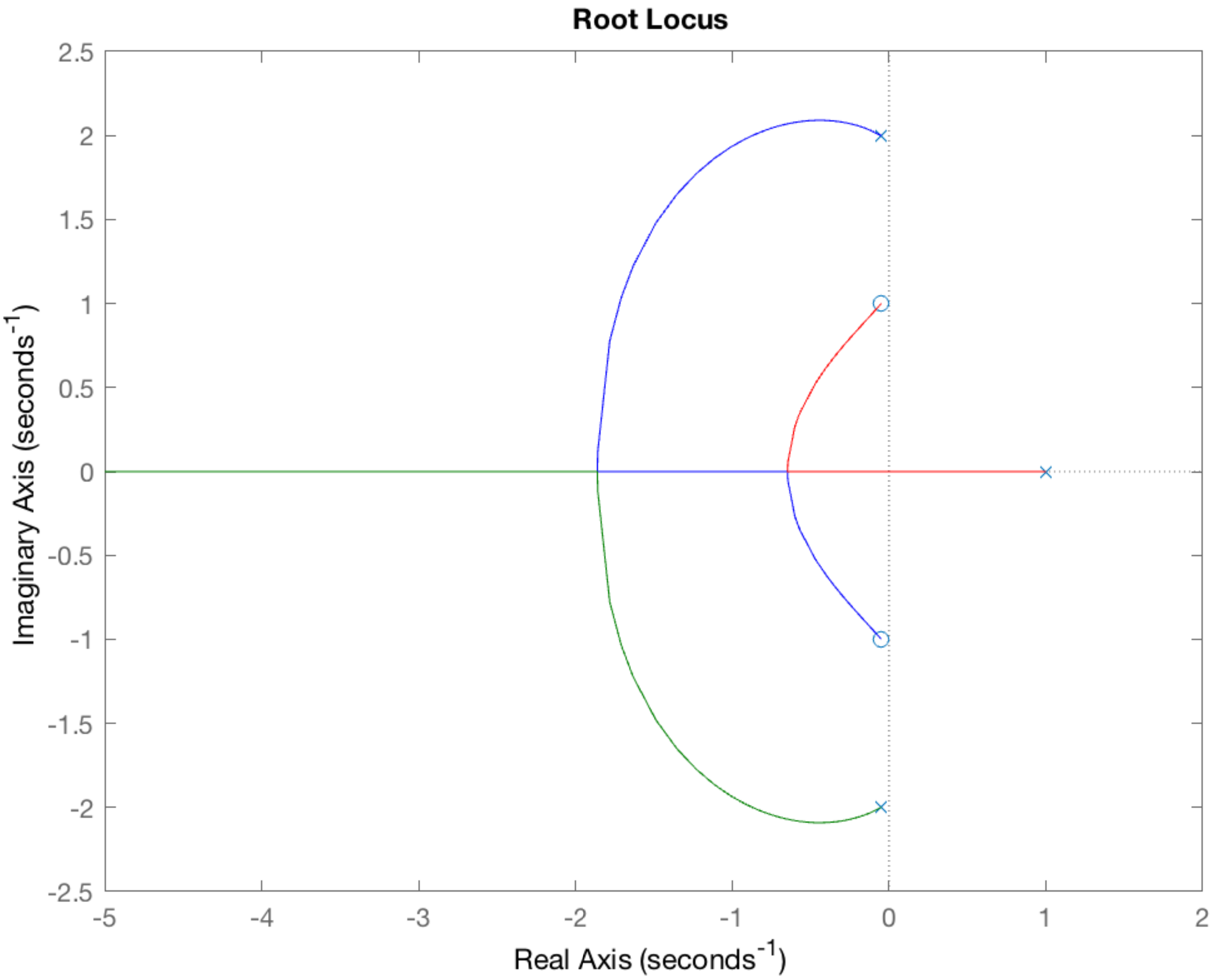
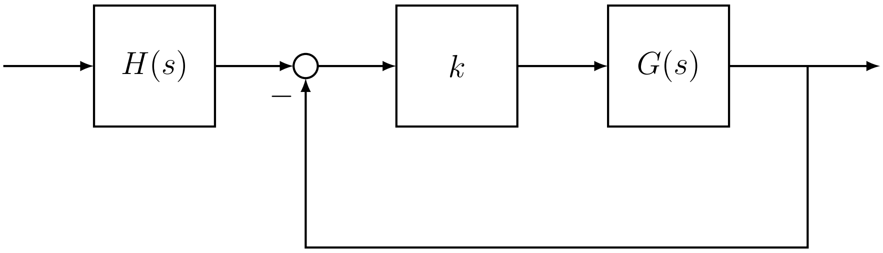


[25%]

(ii) For any stabilising k (we need $k > 4$) let

$$T = \frac{kG}{1 + kG}$$

and define $H = T^{-1}/(s+1)$ in the following block diagram.



[20%]

Examiner's comment. Part (a)(i)B was poorly understood by many candidates, though most completed (a)(ii) and (iii) correctly. Around half of the candidates didn't grasp that the main issue in (a)(iv) was the cancellation of lightly-damped plant poles and zeros. Both parts of Part (b) were generally very well done.

M.C. Smith, 5 May 2024