4F1 Solution 2021

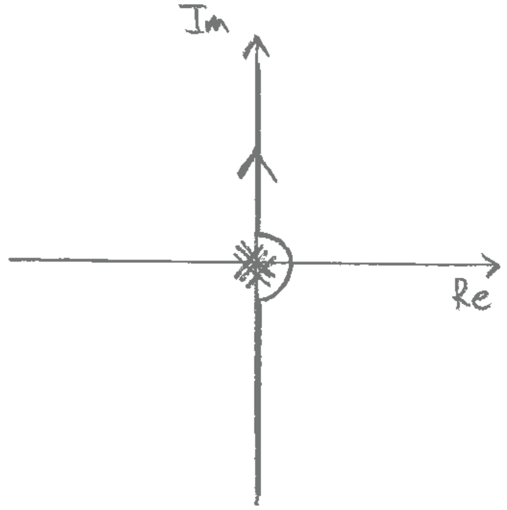
1(a)

Consider the map s→G(s) where G(s) is the transfer function of a system. The root locus diagram is the values of s in the complex plane that map to the negative real line under this map, or equivalently it is the inverse image of the negative real axis. Points on the imaginary axis can be found from the intersection of the Nyquist diagram of G(s) with the negative real line.

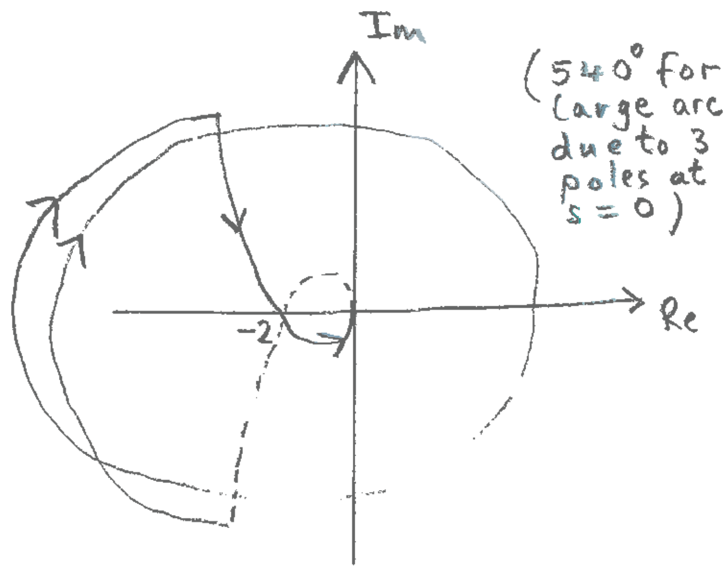
(b)

For a Type 1 system, the steady-state error to a step input is 0, and there is a steady state error to a ramp input. For a Type 2 system the steady-state error to a step or ramp input is 0, and there is a steady-state error to a parabolic input. Feedback systems of type 0 can have a steady state error to a step input. Hence there are improved steady state properties with an increasing N. However, feedback systems of type N where N>0 are open loop unstable and hence problematic behaviour can occur when no feedback is present. Also for large N systems are more difficult to stabilize and they can have a poor performance.

(c)(i)



Nyquist I-contour



Nyquist diagram (not to scale)

(c)(i) cont.

Nyquist stability criterion requires 0 net encirclements of $-1/k$ for closed-loop stability. True if and only if $k > \frac{1}{2}$.

$0 < k < \frac{1}{2}$	2 clockwise encirclements of $-\frac{1}{k}$	2 CL RHP poles
$\frac{1}{2} < k$	0 encirclements " "	0 " " "
$-\infty < k < 0$	1 clockwise " " "	1 " " "

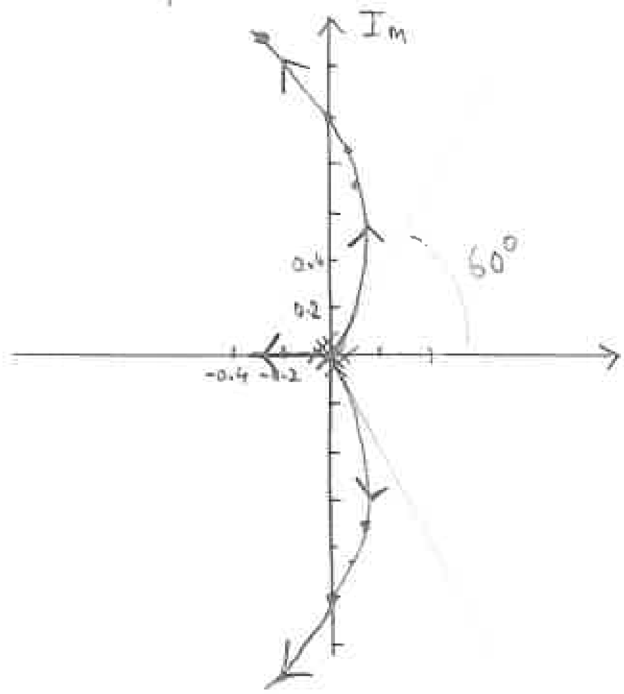
(ii) See attached for hand construction of image of a square grid in the s-plane with spacing 0.1, using the idea of a conformal map.

[Actual transfer function: $G(s) = (s+1)^2/s^3$. True location of closed-loop poles:

k	s
$\frac{1}{4}$	$\cdot 091 \pm j\,0.76$
$\frac{1}{3}$	$\cdot 063 \pm j\,0.85$
$\frac{1}{2}$	$\pm j$
1	$-0.22 \pm j\,1.31$

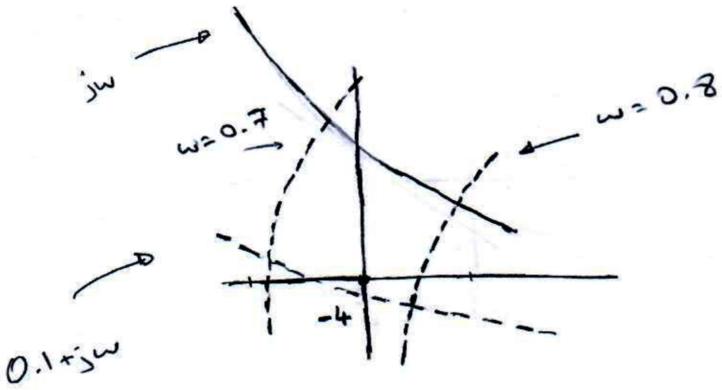
]

(iii)



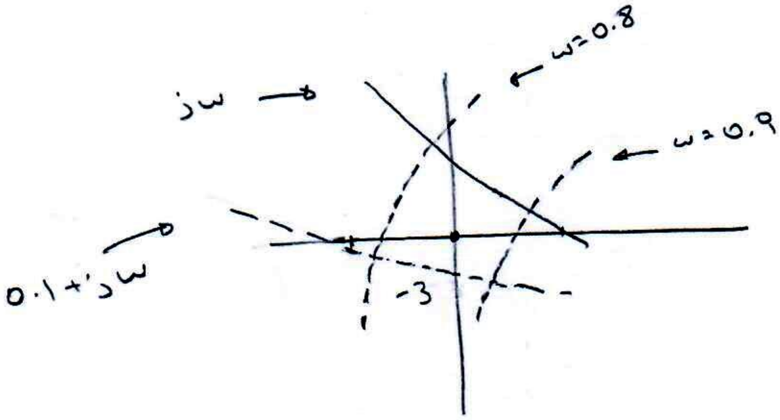
part of root-locus diagram near origin

(c) (ii)



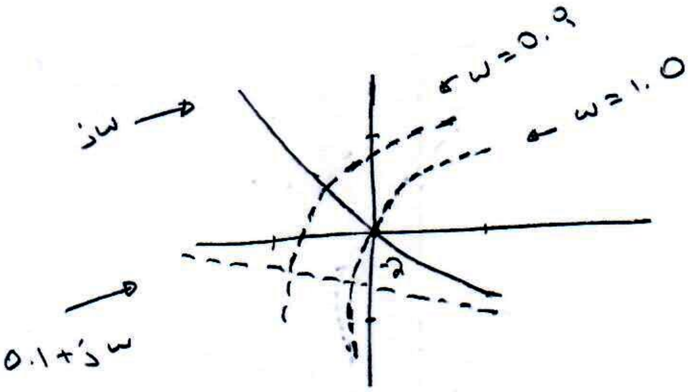
$k = \frac{1}{4}$

$s \approx 0.08 \pm 0.78 j$



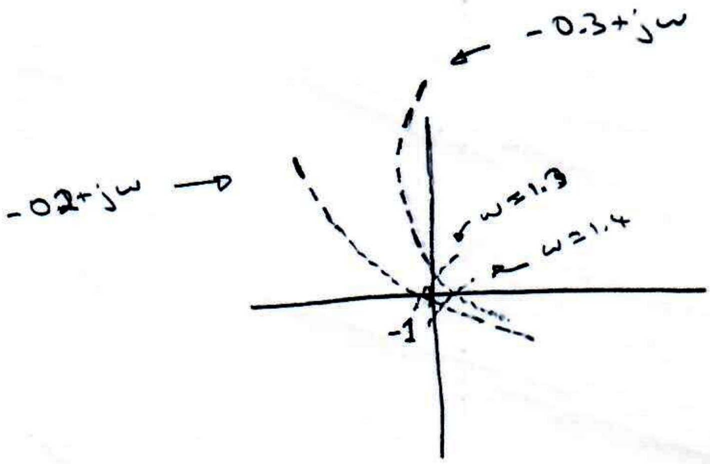
$k = \frac{1}{3}$

$s \approx 0.06 \pm 0.86 j$



$k = \frac{1}{2}$

$s = \pm j$



$k = 1$

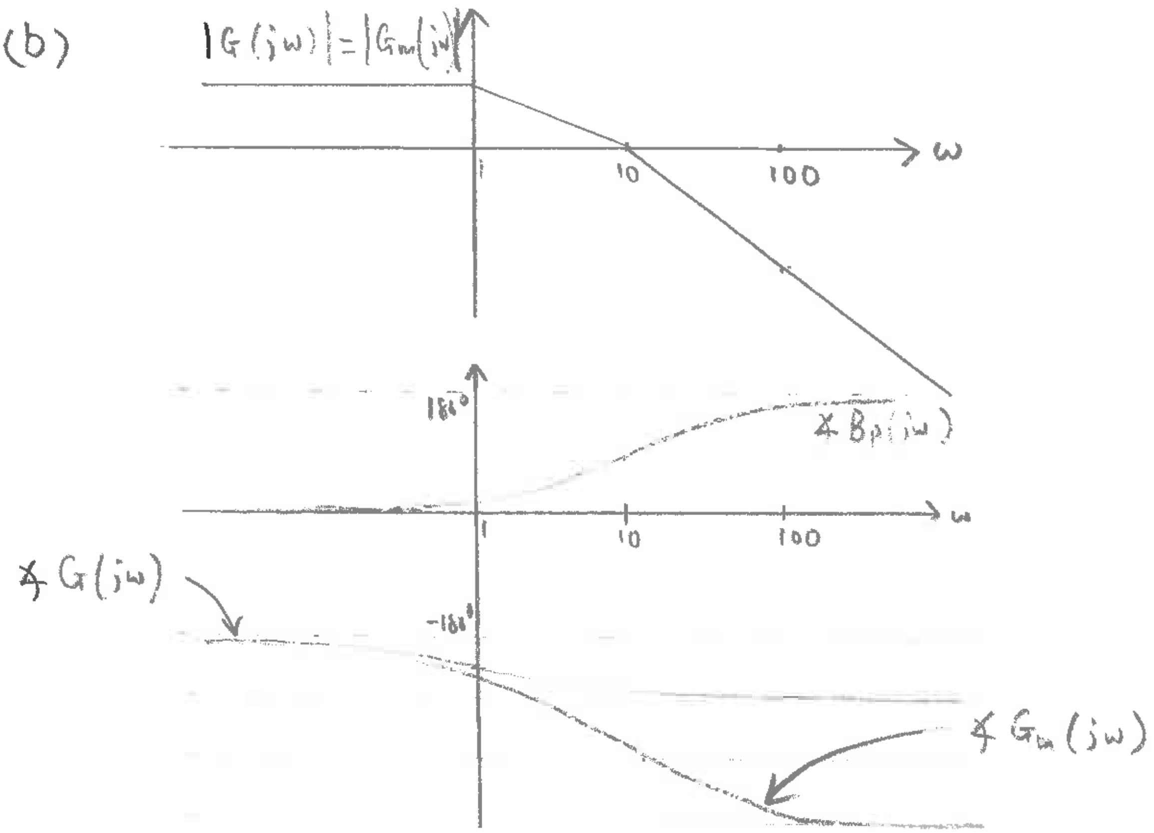
$s \approx -0.22 \pm 1.32 j$

2. (a)

$$G(s) = \frac{1000}{(s+1)(s+10)(s-10)}$$

$$= \underbrace{\frac{-1000}{(s+1)(s+10)^2}}_{G_m} \quad \underbrace{\left(\frac{10+s}{10-s}\right)}_{B_p}$$

(b)



(c) Difficult to achieve a crossover much below 10 rad s^{-1}

(d) Asymptote centre: $(-a+10-10)/3 = -a/3$

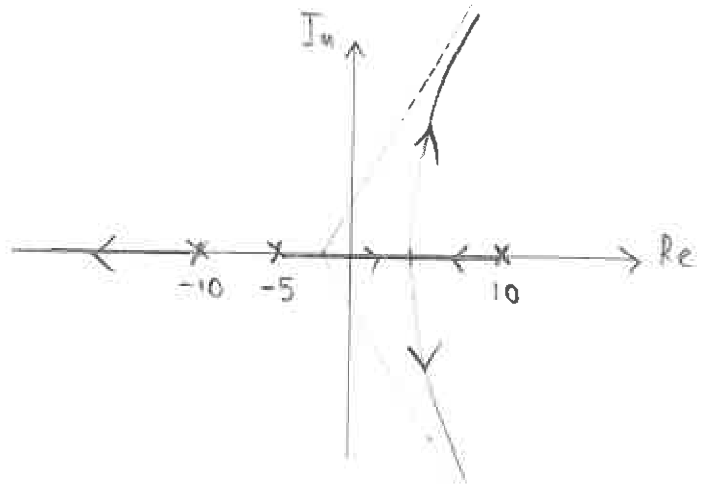
Breakaway points:

$$0 = (s+a)2s + s^2 - 100$$
$$= 3s^2 + 2as - 100$$

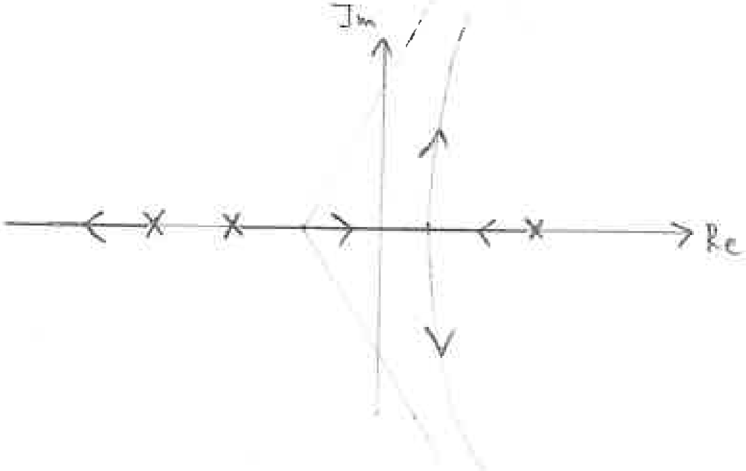
roots: $$\frac{-2a \pm \sqrt{4a^2 + 1200}}{6} = \frac{-a \pm \sqrt{a^2+300}}{3}$$

always one pos. & one neg. root

$a = +5$

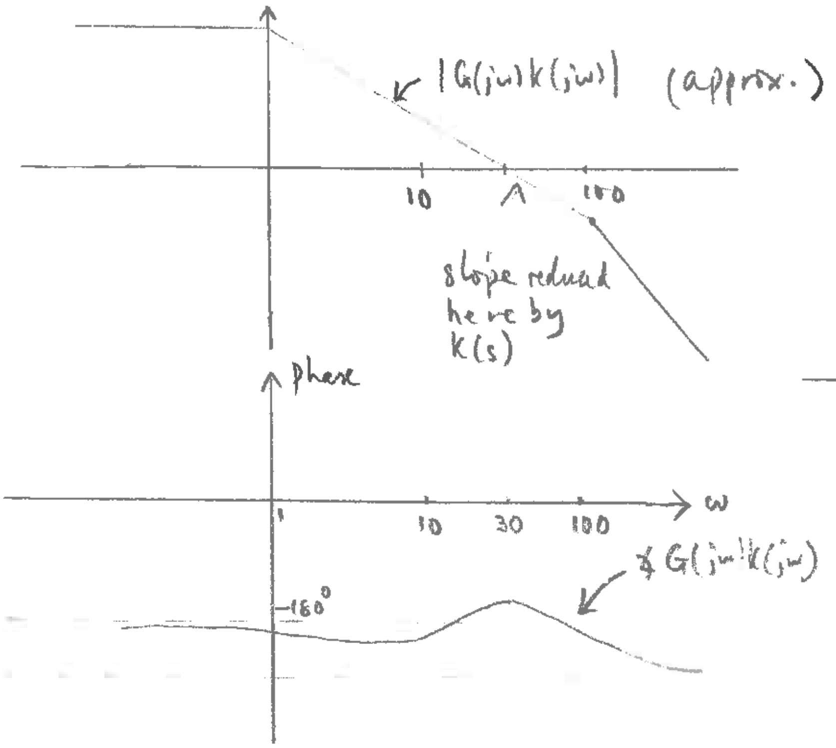


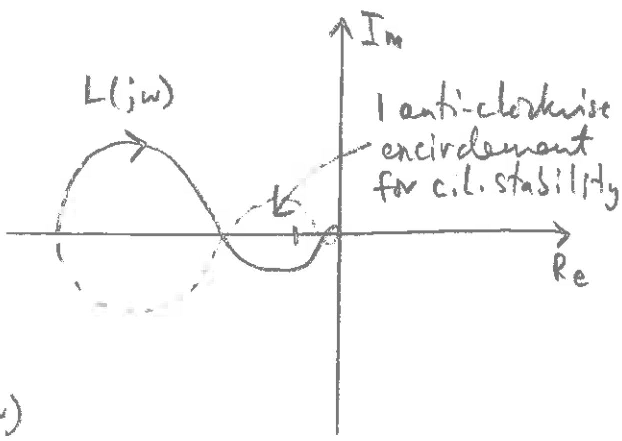
$a = +15$



(e) Follows directly from part (d) since breakaway point is always to the right of the origin the plant is never stabilised.

(f) Choose desired crossover comfortably above $10\ \text{rad s}^{-1}$. E.g. $\omega_c = 30\ \text{rad s}^{-1}$. Need close to 120° lead from $K(s)$. $b = 3.8$ gives $\phi_{max} = 60.5^\circ$ from a single lead compensator. Need $k \simeq 30$ to achieve desired crossover.





$L(j\omega) = G(j\omega)K(j\omega)$

3(a) Stable (min. phase) L with 2^{nd} order roll-off at high freq.

Hence:

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$

$$|S| = \frac{1}{|1+L|} \leq \frac{1}{1-|L|} \quad (\text{for } |L|<1)$$

$$\leq \frac{1}{1-20\omega^{-2}} \quad \text{for } \omega \geq 100$$

Hence:

$$0 \leq \int_0^{\omega_1} \ln 0.4\, d\omega + \int_{\omega_1}^{100} \ln 1.2\, d\omega - \int_{100}^{\infty} \ln(1-20\omega^{-2})\, d\omega$$

$$= \omega_1(-0.916) + (100-\omega_1)\,0.182 \quad - \quad ''$$

Using the hint with $x = 20\omega^{-2}$ and noting that we may take $\varepsilon = 2\times10^{-3}$:

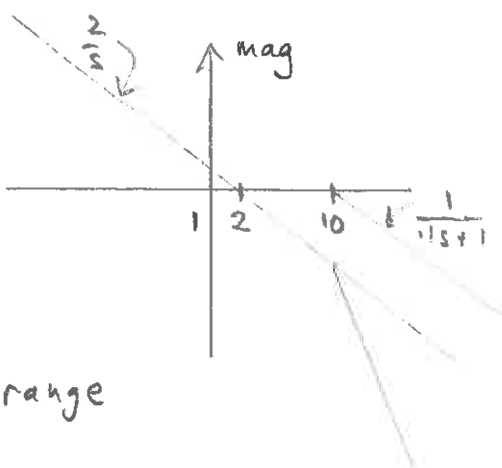
$$0 \leq \quad '' \quad\quad '' \quad + 1.002\int_{100}^{\infty} 20\omega^{-2}\, d\omega$$

$$'' \quad 20.04\left[-\frac{1}{\omega}\right]_{100}^{\infty} = \frac{20.04}{100}$$

$\Rightarrow \; 0 \leq -1.09\,\omega_1 + 18$

$\Rightarrow \; \omega_1 \leq 16.39$ rad/sec

(b)(i) $\quad L(s) = \dfrac{2}{s}\,\dfrac{10}{s+10}$



$|S| \leq 0.4 \iff |1+L| \geq 2.5$

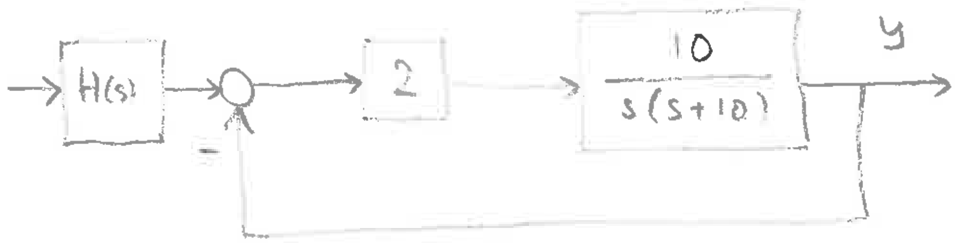
It is sufficient that $|L| \geq 3.5$.

L is approximated by $2/s$ in the low frequency range

$\Rightarrow \omega_1$ (est.) $= 2/3.5 = .57$ rad/sec.

It should be possible to increase ω_1 by careful shaping of $L(s)$ to increase loop gain over a wider frequency range.

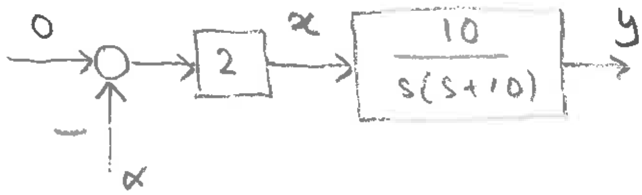
3(b)(ii)



Need: $$H \, \frac{\frac{20}{s(s+10)}}{1 + \frac{20}{s(s+10)}} = \frac{1}{(0.1s+1)^2}$$

$$\Rightarrow \quad H = \frac{s^2 + 10s + 20}{20\,(0.1s+1)^2}$$

(c) Assume a zero reference is applied. A constant value (α) of pitch angle sensor gives the following situation:



with corresponding open-loop dynamics:

$$\ddot{y} + \dot{y} = -20\alpha$$

Particular soln. (neglected fast decaying transient) is:

$$y(t) = -20\alpha t$$

i.e. a ramp response of the pitch angle!

Adequate fault tolerance is needed in the control system, e.g. sensor redundancy. [cf. Boeing 737 Max 8.]

ENGINEERING TRIPOS PART IIB 2021

ASSESSOR'S REPORT, MODULE 4F1

Q1. Root locus and Nyquist diagrams

The question was attempted by about half of the students. Many students had difficulties in sketching the Nyquist diagram in part c(i). There were good attempts for part c(ii) associated with locating closed loop poles close to the imaginary axis, though mistakes were often made in identifying the grid in the Nyquist diagram. Parts (a), (b), c(iii) were generally done well.

Q2. Stabilizing a system with an unstable pole

This was the most popular question and was attempted by most students. There were generally good attempts with common mistakes including wrong signs in the all-pass function, mistakes in the Bode diagrams, and errors in the root locus diagrams.

Q3. Performance limits

The question was attempted by about half of the students. Many students had difficulties in completing part (a) by applying the inequality provided. Most students completed part b(ii) on the design of a 2-degree-of-freedom control system and there were also good attempts for part (c).