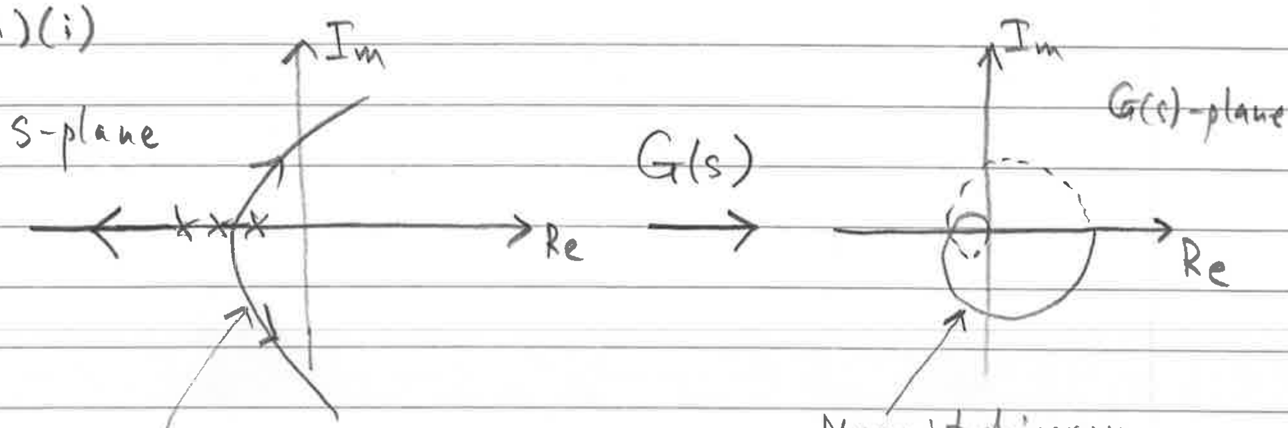
Prof M C Smith

4F1 2014 solutions

1(a)(i)



[15%]

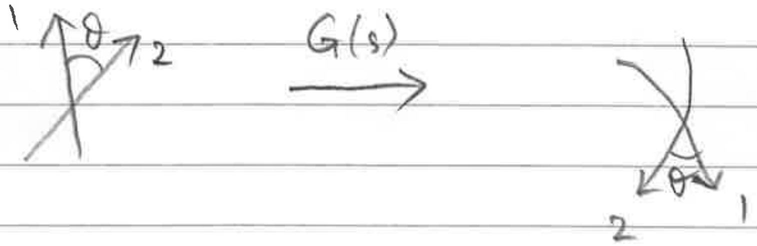
Root-locus
= locus of roots of $\frac{1}{k}+G(s)$ as k varies from 0 to ∞
= inverse image(s) of negative real axis
[namely, the set of points in s-plane which are mapped to negative real axis]

Nyquist diagram
= image of imaginary axis

(ii) $G(s)$ is conformal at points where $\frac{d}{ds}G(s)$ exists and is non-zero

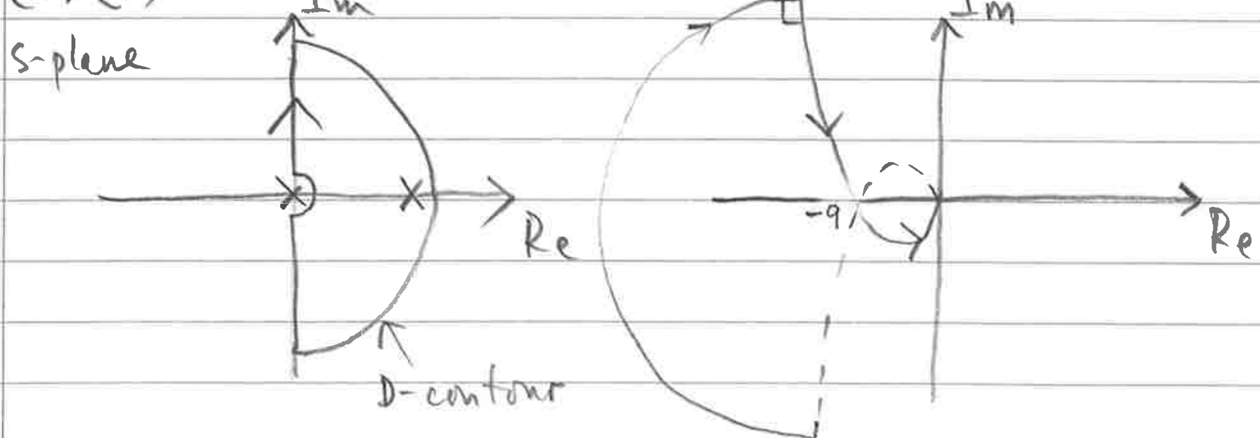
A conformal mapping preserves angle and sense

[15%]



Breakaway points in the root-locus are non-conformal points. Useful in Nyquist diagrams for sense in map of small indentations and for estimating closed-loop poles near imag. axis

(b)(i)



$-\infty < -\frac{1}{k} < -9 \iff 0 < k < \frac{1}{9}$

$\iff$ 1 clockwise encirclement of $-\frac{1}{k}$

$\iff$ 2 closed loop poles in RHP

$-9 < -\frac{1}{k} < 0 \iff \frac{1}{9} < k < \infty$

[20%] $\iff$ 1 counter-clockwise encirclement of $-\frac{1}{k}$

$\iff$ closed Loop stable

= 0 closed loop poles in RHP

$0 < -\frac{1}{k} < \infty \iff -\infty < k < 0$

$\iff$ no encirclements of $-\frac{1}{k}$

$\iff$ 1 closed loop pole in RHP

(ii) See attached for hand method.

Not known for exam: actual $G(s) = \frac{56.7\,(s+6.3)}{s\,(s-6.3)}$

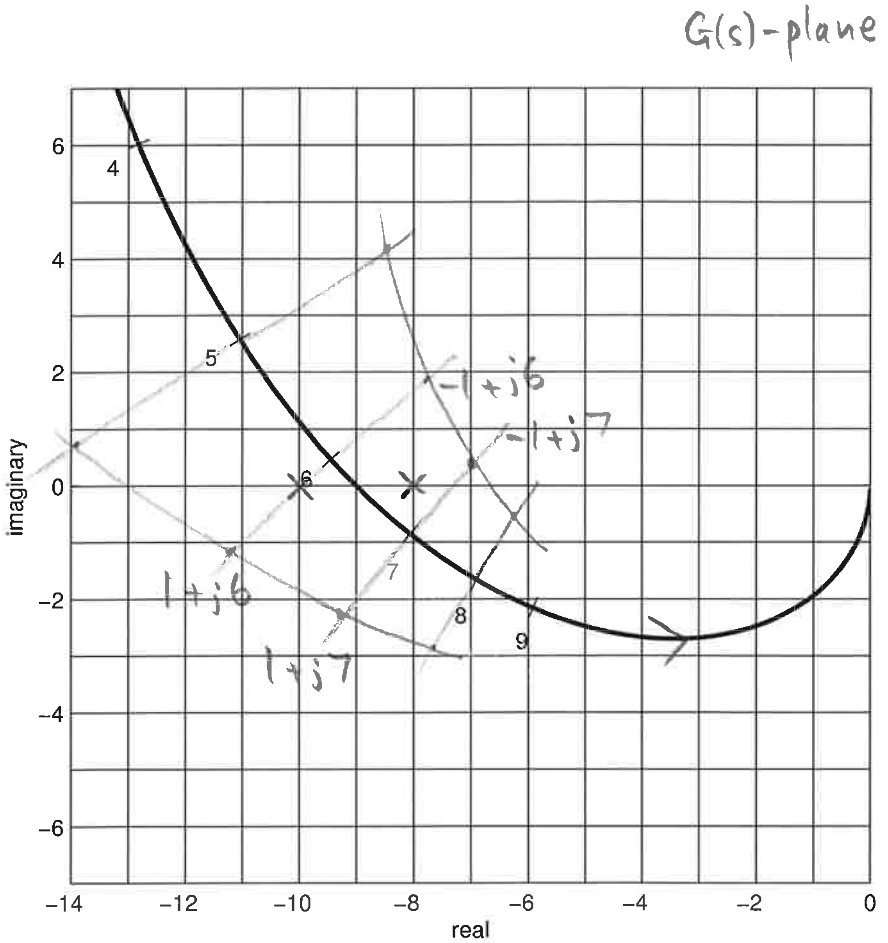
Exact values:

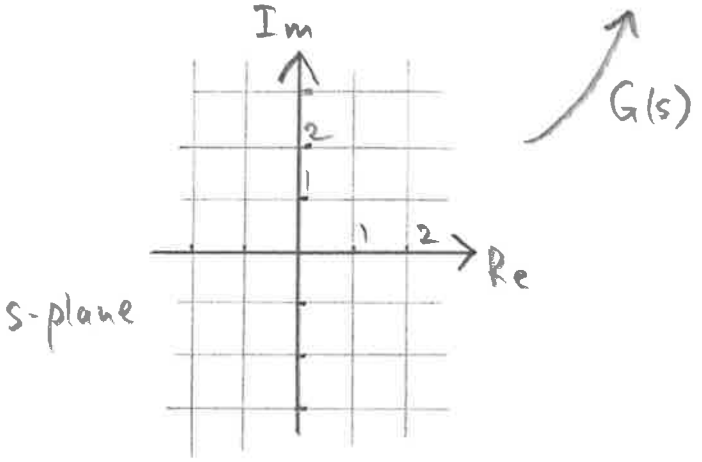
$k = \frac{1}{10}$ $\quad s = 0.315 \pm j\,5.968$

[25%] $k = \frac{1}{9}$ $\quad s = \pm j\,6.3$

$k = \frac{1}{8}$ $\quad s = -0.394 \pm j\,6.670$

(ii) (cont.)



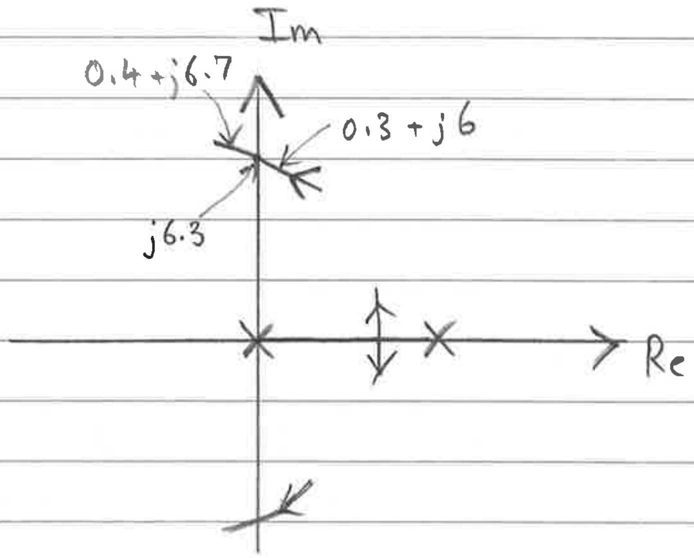


Construct image approximately by ensuring curvilinear squares have sides of close to equal lengths

$k = \frac{1}{8} \Leftrightarrow -\frac{1}{k} = -8 \Leftrightarrow s \simeq -0.4 \pm j6.7$

$k = \frac{1}{10} \Leftrightarrow -\frac{1}{k} = -10 \Leftrightarrow s \simeq 0.3 \pm j6.0$

(iii)



[25%]

Since all poles are in LHP as $k \to +\infty$, any zeros of $G(s)$ must be in LHP

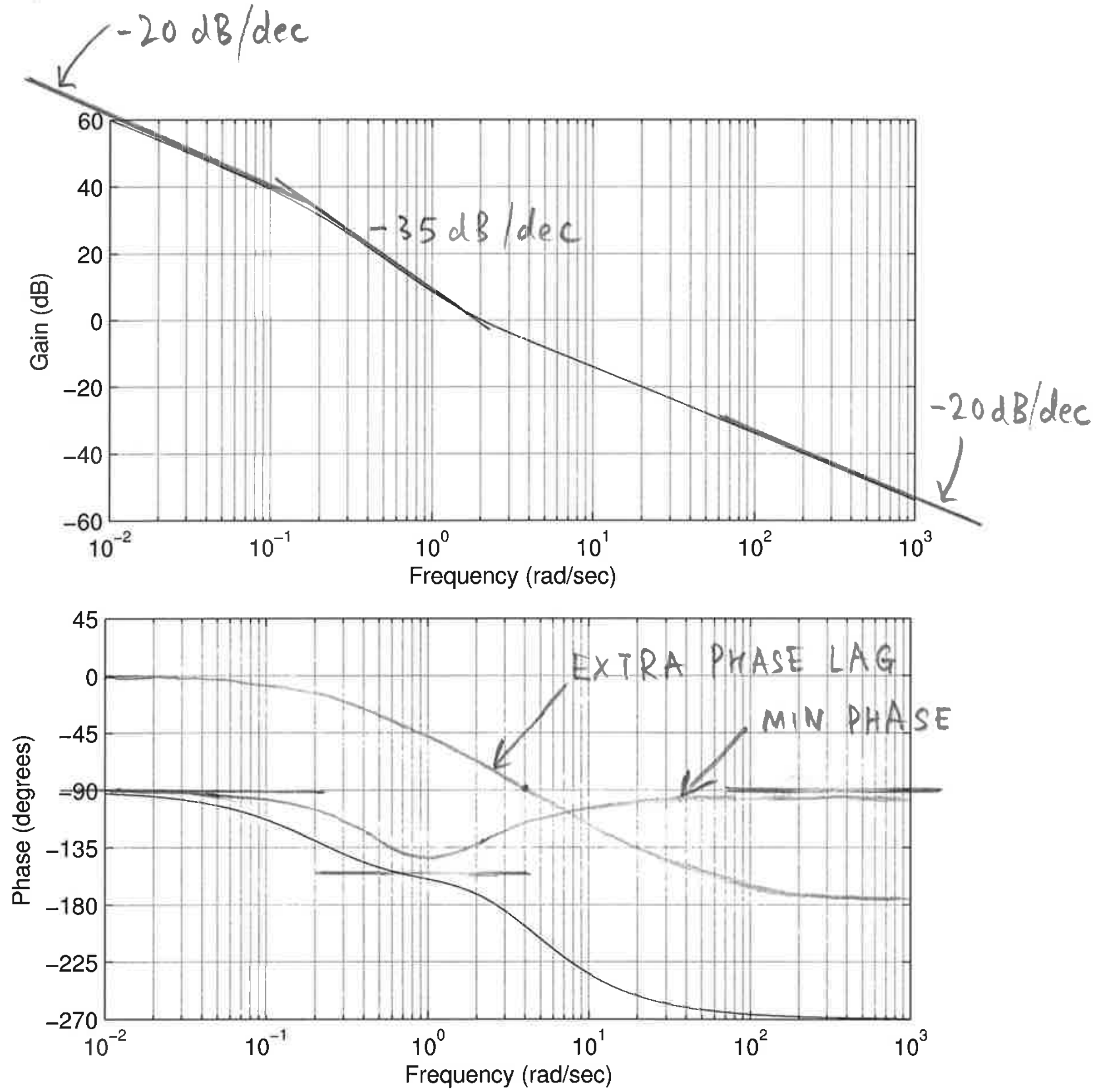
#poles − #zeros = 1 or 2 since otherwise closed-loop can't be stable as $k \to +\infty$ (from the form of the asymptotes)

Q1 Nyquist stability criterion

13 attempts, Average mark 9.8/20, Maximum 17, Minimum 1.

An unpopular question. Close to half of the attempts were very competent on most question parts and achieved good scores. Many of the remainder performed poorly on the bookwork and the application of the Nyquist stability criterion, which together amounted for 50% of the marks, and provided feeble attempts at the last two question parts.

2(a)(i)



(ii) Form of extra phase lag suggests one RHP zero and an all-pass factor

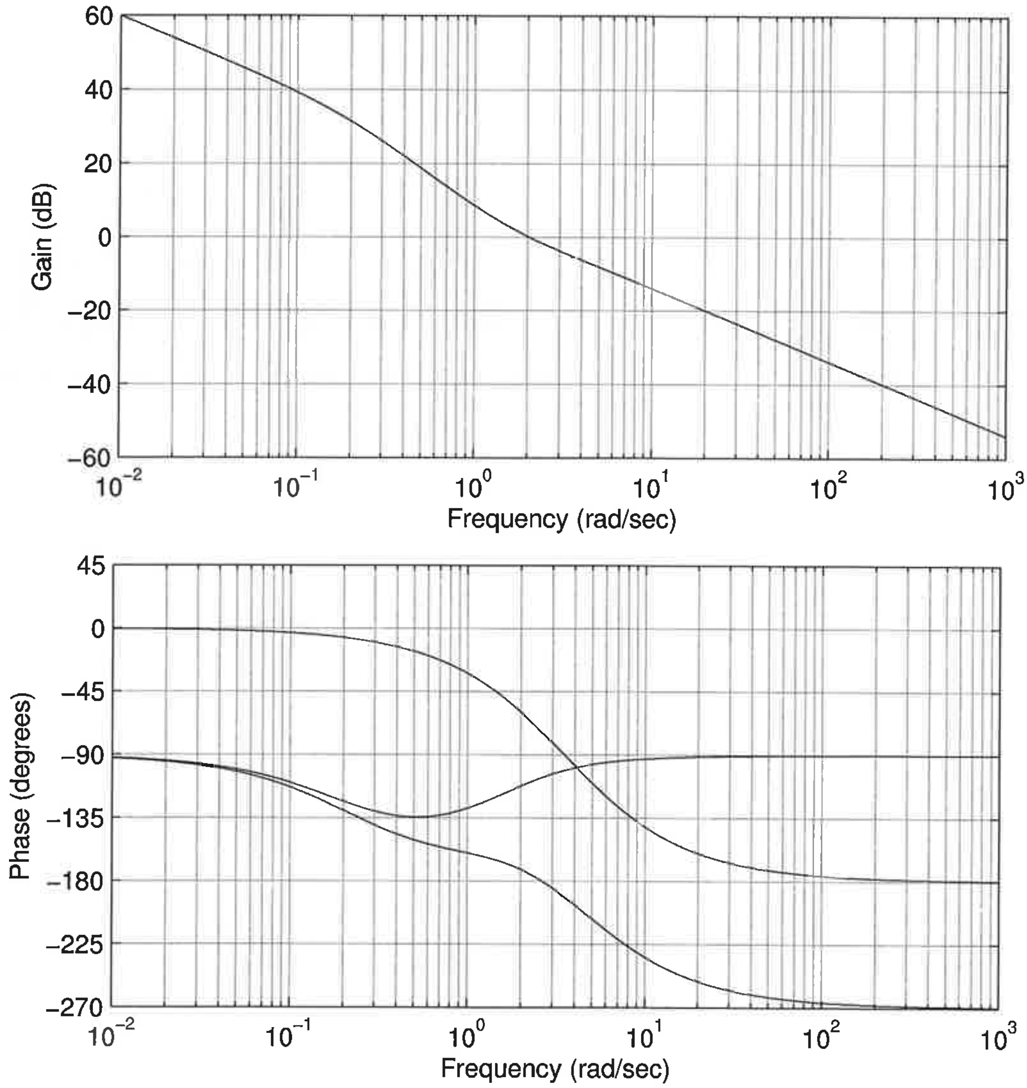
[10%]

$$B(s) = \frac{4-s}{4+s}$$

(iii) Difficult to achieve a gain crossover frequency much greater than 4 rad/s

[10%]

2 (a)(i) Accurate computer plot



Actual all-pass factor: $\dfrac{-s + 3.5}{s + 3.5}$

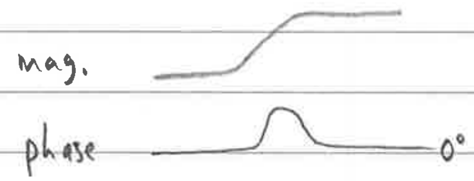
(b)(i) At low freqs. $|G(j\omega)| \sim \dfrac{10}{\omega}$, hence spec A is satisfied already with $k=1$.

At $\omega = 20$, $|G(j\omega)| = 0.1$, hence
~~spec D is satisfied alredy with $k=1$.~~

With $k=1$, $\omega_c = 2$ rad/s, hence spec B o.k.

However PM $\simeq 8^\circ$ so spec C fails

<u>Lead compensator</u>



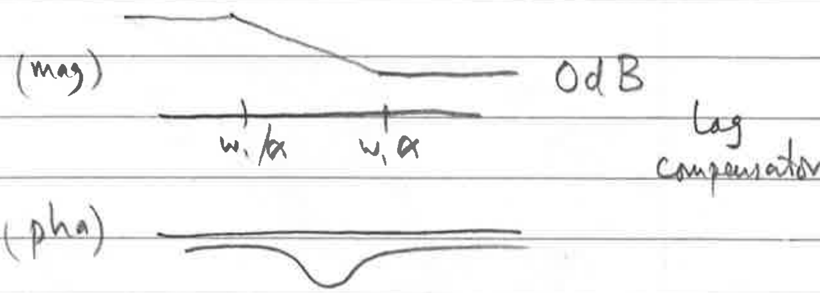
A lead compensator injects
[25%] more gain at high freq.
than low freq. Hence, it will no longer be possible to satisfy both spec A and spec D.

<u>Lag compensator</u>

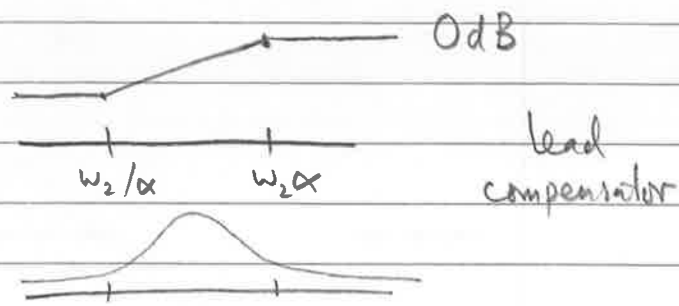
A lag compensator can be used to reduce ω_c, but will decrease phase and PM can't be achieved even with lowest value of $\omega_c = 1$.

(b)(ii)

$$\frac{s + w_1\alpha}{s + w_1/\alpha}$$



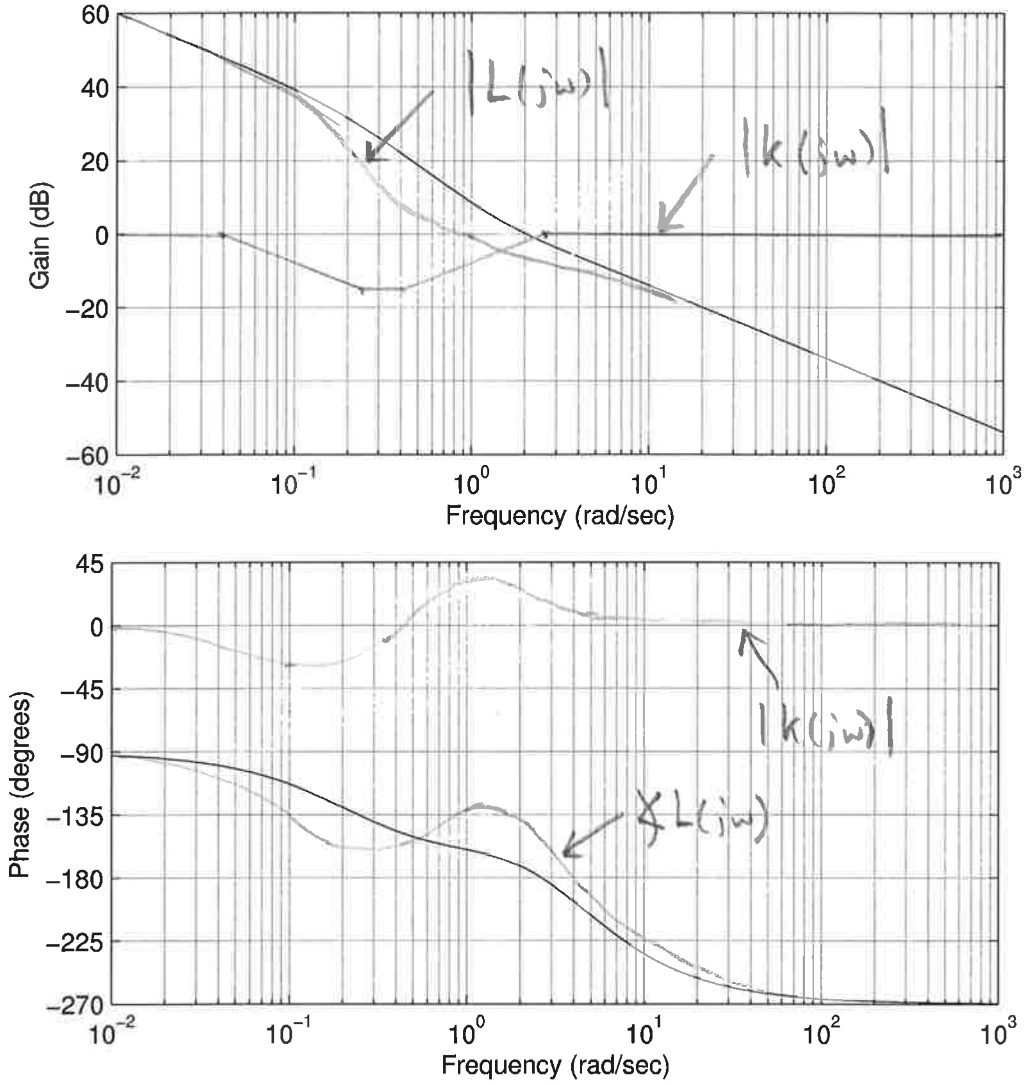
$$\frac{s + w_2/\alpha}{s + w_2\alpha}$$



$k(s)$ has gain 1 (0dB) at low and high freqs. Hence A is satisfied automatically, and also D if $w_2\alpha$ is not too large. To achieve C need lead compensator to increase phase around $\omega = 1$. This suggests $\omega_2 = 1$. If ω_1 is placed a decade lower, the lag compensator has little effect around $\omega = 1$. Hence to achieve $\omega_c = 1$ we need to use the lead to cut the gain by about 8dB at $\omega = 1$. Since lead term has magnitude $1/\alpha$ at ω_2 this suggests $\alpha = 20 \log_{10} 8 \simeq 2.5$, which gives a peak phase of 46° – more than enough to satisfy spec C.

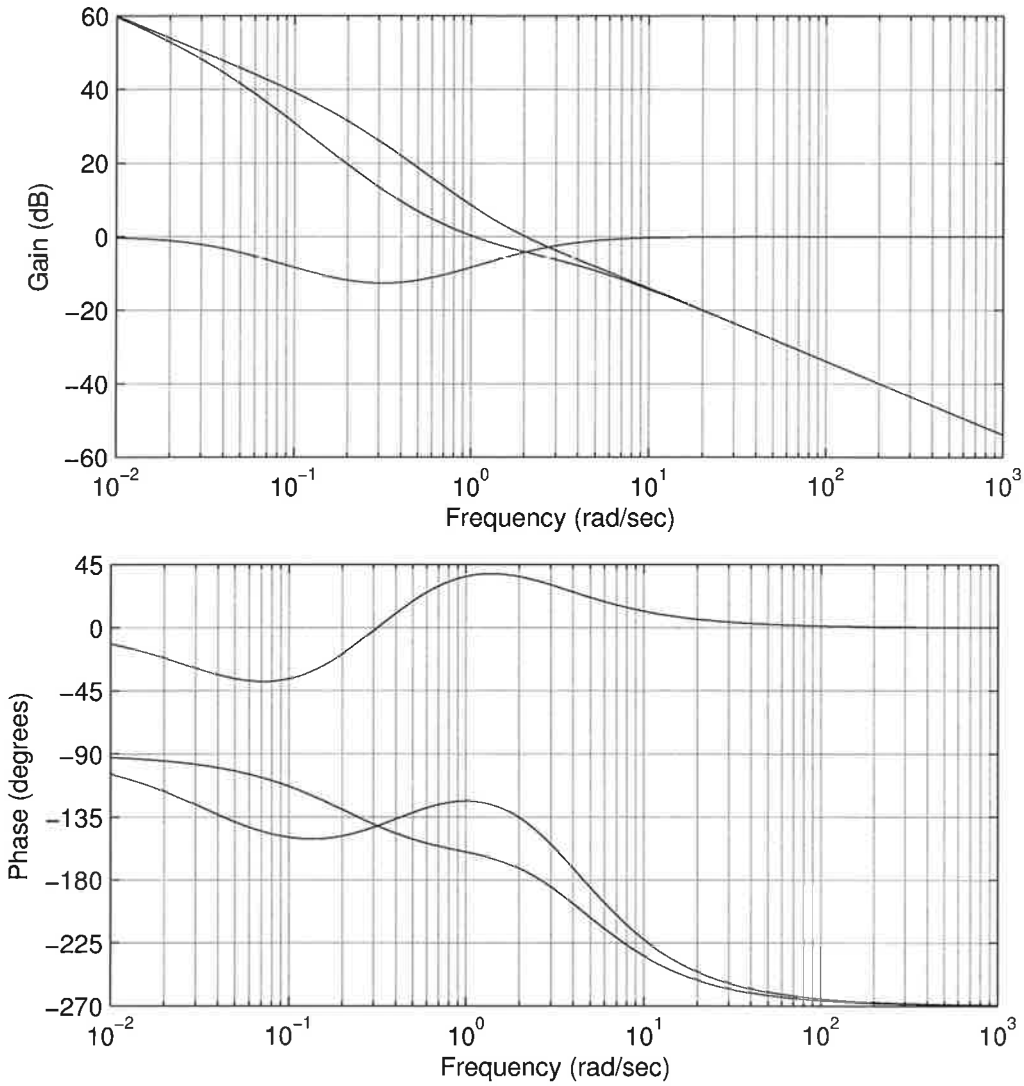
Soln: $\omega_1 = 0.1$, $\omega_2 = 1$, $\alpha = 2.5$

(b)(ii)



Gain (dB)
60
40
20
0
−20
−40
−60
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)
|L(jw)|
|k(jw)|
Phase (degrees)
45
0
−45
−90
−135
−180
−225
−270
Frequency (rad/sec)
|k(jw)|
∡L(jw)

(b)(ii) cont.



Accurate computer plot with:

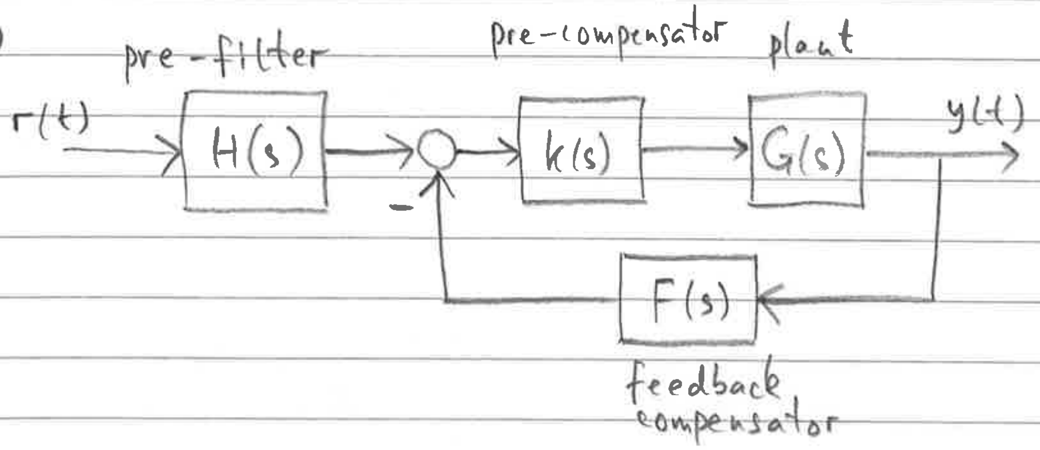
$\omega_1 = 0.1$, $\omega_2 = 1.0$, $\alpha = 2.7$

Q2 Bode gain-phase and compensator design

22 attempts, Average mark 10.9/20, Maximum 19, Minimum 1.

A popular question. The standard part (a) was done well by most candidates. The majority of candidates were on the right lines for Part (b)(i) though few gave completely correct explanations. Part (b)(ii) caused problems with some candidates failing to correctly interpret spec A. Close to half of the candidates gave good solutions but with many of the remainder making little sensible headway.

3 (a)



[15%]

$T_{r\to y}$ must retain RHP zeros of $G(s)$ and roll off at least as fast as $G(s)$ at high frequency

(b)

$$\hat{y}(s) = R(s)\frac{1}{s} = \int_0^\infty y(t)e^{-st}\,dt$$

[15%]

Result follows by putting $s = \alpha$.

(c)(i) Closed-loop poles are the roots of

$$(s^2 - w_1^2)(s+b) + w_1^2 k(s+w_1)$$
$$= (s+w_1)\big((s-w_1)(s+b) + w_1^2 k\big)$$
$$= (s+w_1)\big(s^2 + (b-w_1)s + w_1^2 k - w_1 b\big)$$

[15%]

Closed-loop stable $\Leftrightarrow$ $b > w_1$
$k > b/w_1$

(ii) Choose $b = 3w_1$, $k = 4$

$\Rightarrow$ closed-loop poles: $(s+w_1)(s^2 + 2w_1 s + w_1^2)$
$= (s+w_1)^3$

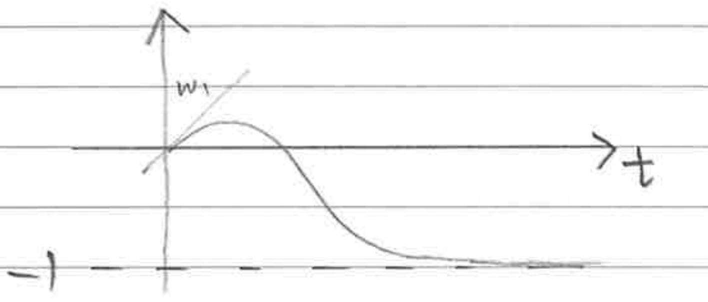
Hence $$\frac{Gk}{1+Gk} = \frac{\dfrac{4\omega_1^2 \cancel{(s+\omega_1)}}{\cancel{(s+\omega_1)(s-\omega_1)(s+3\omega_1)}}}{1 + \dfrac{4\omega_1^2\cancel{(s+\omega_1)}}{\cancel{(s+\omega_1)(s-\omega_1)(s+3\omega_1)}}}$$

[20%] $$= \frac{4\omega_1^2}{(s+\omega_1)^2}$$

Choose $F = 1$ and $H = \dfrac{\omega_1}{4}\dfrac{1}{s+\omega_1}$

(iii) $$T_{r\to u} = \frac{\omega_1^3}{(s+\omega_1)^3}\,\frac{s^2-\omega_1^2}{\omega_1^2}$$

[20%] $$= \frac{\omega_1(s-\omega_1)}{(s+\omega_1)^2}$$



Initial slope $= \lim\limits_{s\to\infty} s\,T_{r\to u}$
$= \omega_1$

Final value $= -\omega_1^2/\omega_1^2$
$= -1$

(iv) $$T_{r\to u} = \frac{kH}{1+GkF}$$

Denominator always has a pole at $s=\omega_1$ by internal stability. If numerator has a pole at $s=\omega_1$ it must occur in k and hence is duplicated in the denominator.

Hence $T_{r \to u}$ has a zero at $s = \omega_1$.

[15%] Part (b) shows that the coil voltage $u(t)$ must reverse sign at least once when there is a step input at $r(t)$.

Q3 2 degree-of-freedom design

23 attempts, Average mark 12.6/20, Maximum 20, Minimum 4.

A popular question which was generally well-answered by most candidates. Many candidates failed to take advantage of the cancelling linear fact in Part (c)(i) which made the working more complicated. Many candidates did not see the quick way to find the required transfer function in Part (c)(ii) and made mistakes with complicated working. Otherwise, candidates showed a good understanding of the topic and there were a number of outstanding solutions.