Version ICL/3

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 28 April 2021 1.30 to 3.10

Module 4F1

CONTROL SYSTEM DESIGN

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 4F1 Formulae sheet (3 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) Explain how the root-locus diagram for a system can be defined in terms of a mapping between complex planes. Hence explain how a Nyquist diagram can be used to find points in the root-locus diagram of a system that are on the imaginary axis. [10%]

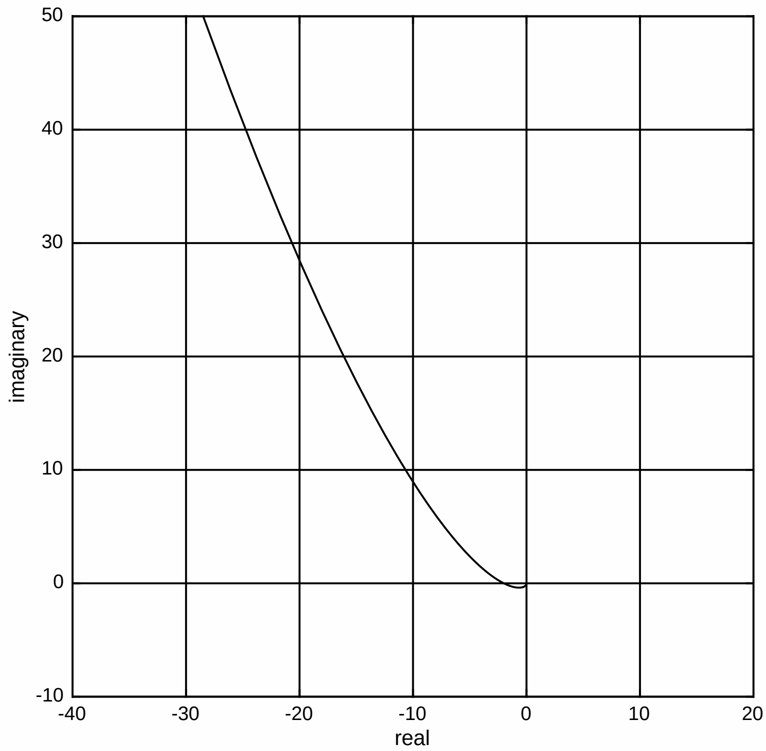
(b) Discuss advantages and disadvantages of type N control systems in terms of the value of N. [10%]

(c) A transfer function $G(s)$ has a pole of multiplicity three at the origin and no poles satisfying $\mathrm{Re}(s) > 0$. The transfer-function gain vanishes at high frequency, i.e. $|G(j\omega)| \to 0$ as $\omega \to \infty$. Figure 1 shows part of the frequency response $G(j\omega)$ as a solid curve at two scales for a range of positive frequencies. The lower plot shows also as dashed curves the image of a square grid in the s-plane under $G(s)$ with $s = \sigma + j\omega$ where $\omega = 0.6, 0.7, \ldots, 1.5$ rad s^{-1} and $\sigma = \pm 0.1, \pm 0.2, \pm 0.3$. These curves are not labelled in the diagram, but it is possible from the information given to correctly identify them. Constant gain feedback k is applied to the system with the negative feedback convention.

(i) Sketch the complete Nyquist diagram for $G(s)$ and use it to determine the number of poles of the closed-loop system in the right half-plane for all values of k both positive and negative. [25%]

(ii) Use Fig. 1 to estimate the position of any closed-loop poles close to the imaginary axis for k =1/4, k =1/3, k =1/2 and k =1. Illustrate your calculations with a simple sketch, for each value of k, of *the relevant small portion* of the lower Fig. 1 with σ and ω values marked for the bounding curves. [35%]

(iii) Make use of the information on $G(s)$ and your answers to (c)(ii) to provide a sketch of the root-locus diagram for k with $0 \leq k \leq 1$. [20%]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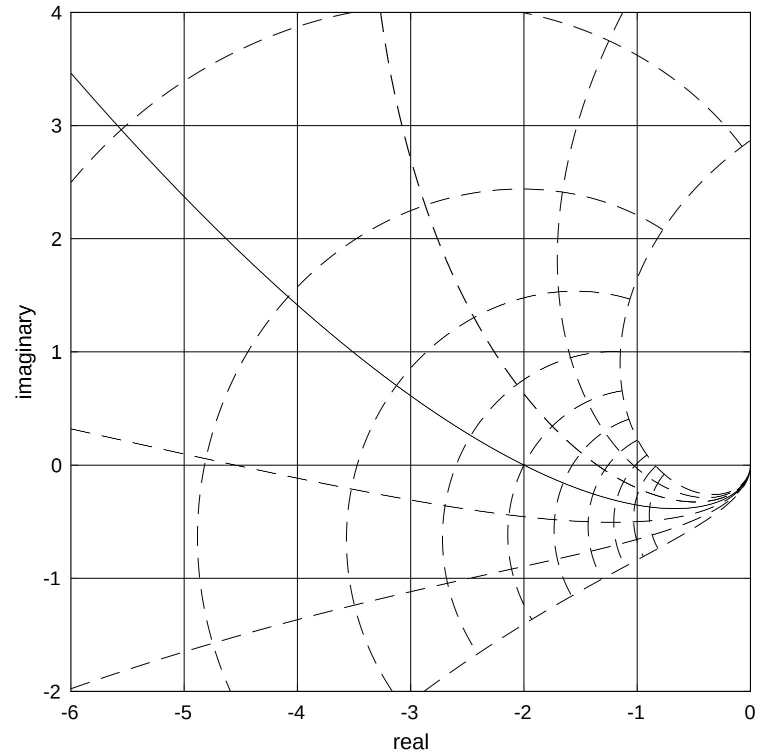
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imaginary
real

Fig. 1

2 A controller is to be designed for a magnetically levitated train. The transfer function of a single superconducting magnet system from coil voltage to nominal height above the track is given by

$$G(s) = \frac{1000}{(s+1)(s^2-100)}\,. \qquad (1)$$

(a) Express the transfer function in the form

$$G(s) = G_m(s)B_p(s)$$

where $B_p(s)$ is a pole-type all-pass function normalised so that $B_p(0) = 1$ and $G_m(s)$ has no poles or zeros with $\mathrm{Re}(s) > 0$. [15%]

(b) Sketch the Bode diagrams of $G(s)$, $G_m(s)$ and $B_p(s)$ on a single plot. [15%]

(c) Comment briefly on any limitations that may be experienced in the design of a controller for $G(s)$. [10%]

(d) Sketch the root-locus diagram for positive gain k for the plant

$$G(s) = \frac{1}{(s+a)(s^2-100)}$$

where $a > 0$ is a positive constant. You may find it useful to do two sketches according to whether $a < 10$ or $a > 10$. Pay particular attention to the location of any breakaway point. [25%]

(e) Hence, or otherwise, deduce that the plant (1) cannot be stabilised by a first order compensator of the form [10%]

$$K(s) = \frac{k(s+1)}{s+a}$$

(f) A double lead compensator is proposed in the form

$$K(s) = k\left(\frac{bs+\omega_c}{s+b\omega_c}\right)^2$$

where $b > 1$. Select suitable parameters k, b and ω_c that can stabilise $G(s)$ in the standard unity gain negative feedback configuration to achieve a phase margin of at least 30°. Justify that your design is stabilising by means of a Nyquist diagram sketch of $G(s)K(s)$. [25%]

3 A pitch control system is to be designed for an aircraft. A simple model relating the pitch angle $y(t)$ to the elevator angle $x(t)$ is given by

$$\ddot{y}(t) + c\dot{y}(t) = bx(t)$$

where c is a coefficient of aerodynamic damping and b is a coefficient of effectiveness of the elevator. It may be assumed that $c = 10\ \mathrm{s}^{-1}$ and $b = 10\ \mathrm{s}^{-2}$.

(a) It is desired that the sensitivity of the control system is reduced over as wide a frequency range as possible. Plant uncertainty suggests that an upper bound on the magnitude of the return ratio is specified at high frequency. The following specifications are proposed, where $S(s)$ denotes the sensitivity function and $L(s)$ denotes the return ratio:

A: $|S(j\omega)| \leq 0.4$ for $0 \leq \omega \leq \omega_1$;

B: $|S(j\omega)| \leq 1.2$ for all ω;

C: $|L(j\omega)| \leq 20/\omega^2$ for $\omega \geq 100\ \mathrm{rad\,s}^{-1}$.

Find an upper bound on the achievable ω_1. State clearly but do not prove any results you use. [Hint. You may assume the inequality: $\ln(1-x) \geq -(1+\epsilon)x$ providing $0 < x < \epsilon < 1/2$.] [35%]

(b) Let $G(s)$ denote the transfer-function $T_{\bar{x}\to\bar{y}}$ and suppose $L(s) = 2G(s)$. You may assume that specification B in (a) is achieved.

(i) Estimate the value of ω_1 for specification A in (a) that the design achieves and comment on whether this design could be improved. [20%]

(ii) Design a 2-degree-of-freedom control system to achieve a transfer function from reference input to pitch angle equal to [20%]

$$\frac{1}{(0.1s+1)^2}.$$

(c) Suppose that the aircraft is equiped with a single pitch angle sensor which malfunctions in flight, providing only a constant non-zero value to the flight control computer. What effect will this have on the aircraft? What advice would you give to the manufacturer to mitigate this problem? [25%]

END OF PAPER

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Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

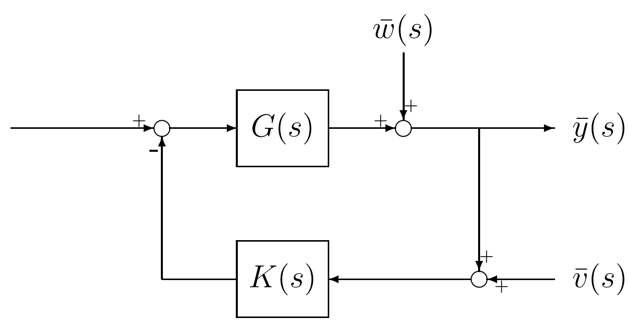
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\text{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\, dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

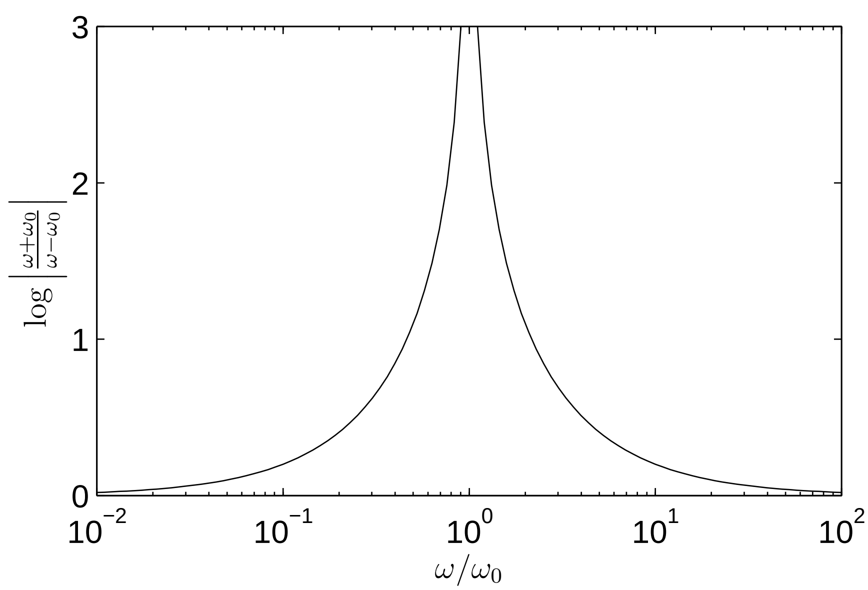


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2} \left.\frac{d\log|L(j\omega_0 e^v)|}{dv}\right|_{v=0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\big(j|s_0|e^{v}\big)\right| dv}$$

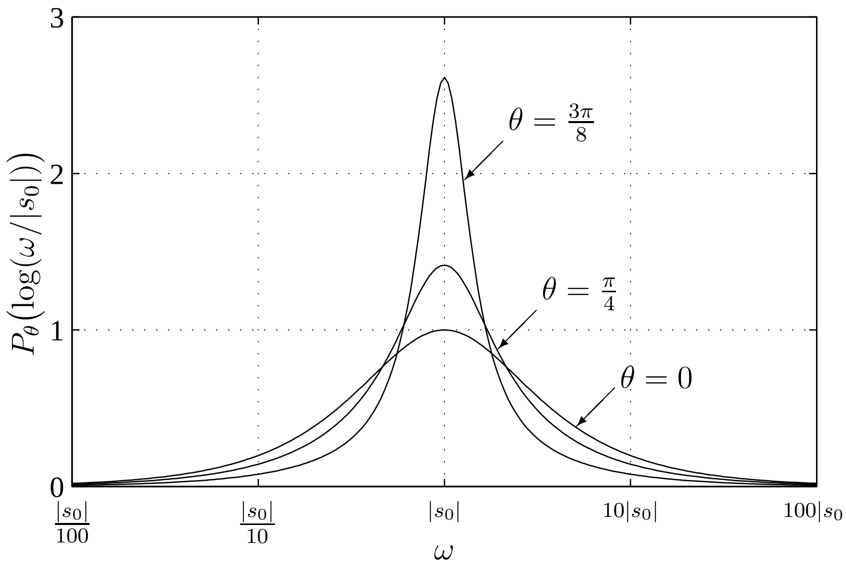
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith
November 2020

Engineering Tripos Part IIB
2021
Paper 4F1: Control System Design
Answers

1. (c)(i) $0 < k < 1/2$, 2 RHP poles; $k > 1/2$, 0 RHP poles; $-\infty < k < 0$, 1 RHP pole

(c)(ii) poles: $0.091 \pm 0.76j,\ 0.063 \pm 0.85j,\ \pm j,\ -0.22 \pm 1.31j$

3. (a) $\omega_1 \leq 16.39$ rad s^{-1}

(b)(i) $\omega_1 \approx 0.57$ rad s^{-1}

(b)(ii) $H = \dfrac{s^2 + 10s + 20}{20(0.1s + 1)^2}$