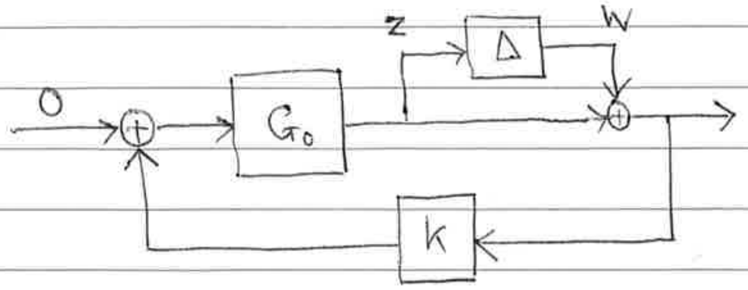
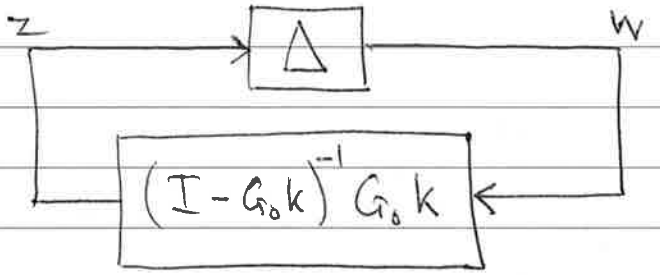
4F2 crib. (2014)   Prof R Sepulchre

1 (a)



$$z = G_0\left(K(w+z)\right) \Rightarrow z = (I - G_0 K)^{-1} G_0 K w$$

Re-write block diagram:



[25%]

By small gain theorem, closed loop is internally stable for all $\|\Delta(s)\|_\infty < \varepsilon$ if and only if $\|(I-G_0 k)^{-1} G_0 k\|_\infty \le \frac{1}{\varepsilon}$

(b) (i)

$$I + G_0 = \frac{1}{s^2+a^2}\begin{bmatrix} s^2+s & a(s+1) \\ -a(s+1) & s^2+s \end{bmatrix} = \frac{s+1}{s^2+a^2}\begin{bmatrix} s & a \\ -a & s \end{bmatrix}$$

$$\Rightarrow S = (I+G_0)^{-1} = \frac{\cancel{s^2+a^2}}{s+1} \cdot \frac{1}{\cancel{s^2+a^2}}\begin{bmatrix} s & -a \\ a & s \end{bmatrix} = \frac{1}{s+1}\begin{bmatrix} s & -a \\ a & s \end{bmatrix}$$

[20%] $$T = I - S = \frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

(ii) Internal stability requires

$$\begin{pmatrix} I & -K \\ -G_0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & I \\ -G_0 & I \end{pmatrix}^{-1} = \begin{pmatrix} (I+G_0)^{-1} & -(I+G_0)^{-1} \\ G_0(I+G_0)^{-1} & (I+G_0)^{-1} \end{pmatrix} = \begin{pmatrix} S & -S \\ T & S \end{pmatrix}$$

[20%] to be in H_∞, which it is by inspection from (i).

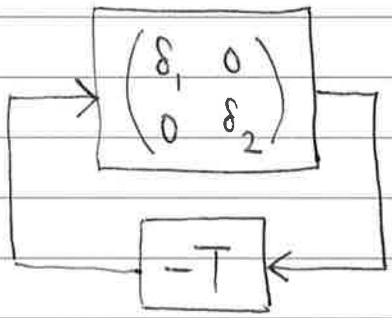
(iii) $(I - G_0 K)^{-1} G_0 K = -(I + G_0)^{-1} G_0 = -T$

$$\|T\|_\infty = \Bigg\| \underbrace{\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}}_{= A} \Bigg\|$$

$$A^* A = \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix} = \begin{pmatrix} 1+a^2 & 0 \\ 0 & 1+a^2 \end{pmatrix}$$

[20%] $\Rightarrow \|T\|_\infty = \sqrt{1+a^2} \Rightarrow \varepsilon_{max} = \dfrac{1}{\sqrt{1+a^2}}$

(iv)



Loop is internally stable for all $\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}$ with $\|\Delta\|_\infty < \frac{1}{\gamma}$ if and only if

$$\sup_\omega \mu_\Delta(T(j\omega)) \leq \gamma$$

where $\mu_\Delta(M) = \dfrac{1}{\min\{\bar{\sigma}(\Delta) : \Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \det(I - M\Delta) = 0\}}$ is the

[15%] structured singular value.

Q1 Multiplicative uncertainty

25 attempts, Average mark 11/20, Maximum 20, Minimum 1.

Many candidates were let down by a lack of facility in doing elementary matrix manipulations. In Part (a) a common mistake was to write the uncertainty block at the input side of the nominal plant rather than the output side. In Part (b)(i) many candidates failed to correctly add the identity matrix to G0(s), and many failed to spot simple cancelling factors.

Q2 H-infinity loop-shaping procedure

19 attempts, Average mark 10/20, Maximum 19, Minimum 2.

Parts (a) and (b) were bookwork. Many candidates were rather imprecise and lost marks as a result. Parts (c)(i) and (c)(ii) required the use of some basic singular value inequalities, and although there were a good number of correct solutions, too many candidates tried to force through to the required bound with a sequence of incorrect steps. Part (c)(iii) was generally well done, though not many candidates made use of the derived bounds as well as their knowledge of the loop-shaping procedure.

2 (a)(i) Suppose $G(s)$ is a $p \times m$ transfer function matrix. $G = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorisation over $\mathcal{H}_\infty$ if $\tilde{M}$, $\tilde{N}$ are matrices with entries in $\mathcal{H}_\infty$ and if

$$\operatorname{rank}\left[\tilde{N}(s) \quad \tilde{M}(s)\right] = p \quad \text{for all } s \text{ with } \operatorname{Re}(s) \geq 0 \text{ or } s = \infty$$

The factorisation is normalised if

[15%] $$\tilde{M}(j\omega)\tilde{M}(j\omega)^* + \tilde{N}(j\omega)\tilde{N}(j\omega)^* = I$$

for all ω.

(ii) Let $X = \begin{bmatrix} K \\ I \end{bmatrix}(I - GK)^{-1}\tilde{M}^{-1}$.

Since $\begin{bmatrix}\tilde{M} & \tilde{N}\end{bmatrix}\begin{bmatrix}\tilde{M}^* \\ \tilde{N}^*\end{bmatrix} = I$ we have

$$\lambda_i(XX^*) = \lambda_i\left(X\left[\tilde{M}\ \tilde{N}\right]\begin{bmatrix}\tilde{M}^* \\ \tilde{N}^*\end{bmatrix}X^*\right)$$ hence

[15%] $\bar{\sigma}(X) = \bar{\sigma}\left(X\left[\tilde{M}\ \tilde{N}\right]\right)$ from which the result follows.

(b) Let $b(G,K)$ denote the inverse of the expression in equation (1). The steps of the $\mathcal{H}_\infty$ loop shaping procedure are:

(1) Scale inputs and outputs to have similar magnitudes.

(2) Plot singular values of $G(j\omega)$.

(3) Choose a weighting function $W(j\omega)$ to shape the singular values of $G(j\omega)W(j\omega)$ as in classical loop shaping: high gain for reduced sensitivity, low gain where large uncertainty and noise.

(4) Design K to maximise $b(GW, K)$ to give K_∞. Iterate to achieve $b(GW, K_\infty)$ greater than about 0.2.

[20%] (5) Set $K = W K_\infty$.

(c) (i) From the hint: $\bar{\sigma}((I-GK)^{-1})$ and $\bar{\sigma}(K(I-GK)^{-1})$ are less than

$$\bar{\sigma}\left(\begin{pmatrix} K \\ I \end{pmatrix}(I-GK)^{-1}\right)$$

$$\bar{\sigma}\left(\begin{pmatrix} K \\ I \end{pmatrix}(I-GK)^{-1}\right) = \bar{\sigma}\left(\begin{pmatrix} K \\ I \end{pmatrix}(I-GK)^{-1}\tilde{M}^{-1}\tilde{M}\right)$$

[15%]

$$\leq \bar{\sigma}\left(\begin{pmatrix} K \\ I \end{pmatrix}(I-GK)^{-1}\tilde{M}^{-1}\right)\bar{\sigma}(\tilde{M})$$

$$\leq \gamma\,\bar{\sigma}(\tilde{M})$$

(ii)

$$\bar{\sigma}\left(\begin{bmatrix} K \\ I \end{bmatrix}(I-GK)^{-1}G\right) = \bar{\sigma}\left(\begin{bmatrix} K \\ I \end{bmatrix}(I-GK)^{-1}\tilde{M}^{-1}\tilde{N}\right)$$

$$\leq \bar{\sigma}\left(\begin{bmatrix} K \\ I \end{bmatrix}(I-GK)^{-1}\tilde{M}^{-1}\right)\bar{\sigma}(\tilde{N})$$

[15%]

$$\leq \gamma\,\bar{\sigma}(\tilde{N})$$

and the result follows from the hint.

(iii) (1) $\gamma = 2.5$. Since $\bar{\sigma}(\tilde{N}), \bar{\sigma}(\tilde{M}) \leq 1$ all closed loop transfer functions have H_∞ norms no greater than 2.5 – only moderate worst case amplification. Corresponds to a robustness margin of 0.4 – an acceptable value.

(2) $\gamma = 100$. Corresponds to a robustness margin of 0.01 – too small. Since

[20%]

$$\left\|\begin{bmatrix} K \\ I \end{bmatrix}(I-GK)^{-1}\begin{bmatrix} I & G \end{bmatrix}\right\|_\infty = 100$$

some closed-loop transmission paths will have large amplification – not acceptable.

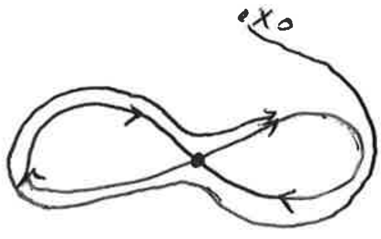
3 -

a) Denote by $\phi(t, x_0)$ the solution of $\dot{x} = f(x)$ at time t for the initial condition $\phi(0, x_0) = x_0$. Then x belongs to the ω-limit set of x_0, denoted by $\omega(x_0)$, if there exists an increasing and unbounded sequence of times $(t_k)_{k \geq 1}$ such that $\phi(t_k, x_0) \longrightarrow x$ as $k \to +\infty$. (10%)

Properties: $\omega(x_0)$ is closed, connected, and invariant. (5%)

(b) (i) If $\omega(x_0)$ does not contain a fixed point, then it is a closed orbit. (15%)

(ii)



The entire ∞-shape is the ω-limit set of x_0. (15%)

(c) The stable manifold of a saddle point $\bar{x}$ is the set of initial conditions that have $\bar{x}$ as an ω-limit set. (10%)

In the phase portrait of a bistable model, a typical situation is that the stable manifold of the saddle point is the separatrix of the two basins of attraction.

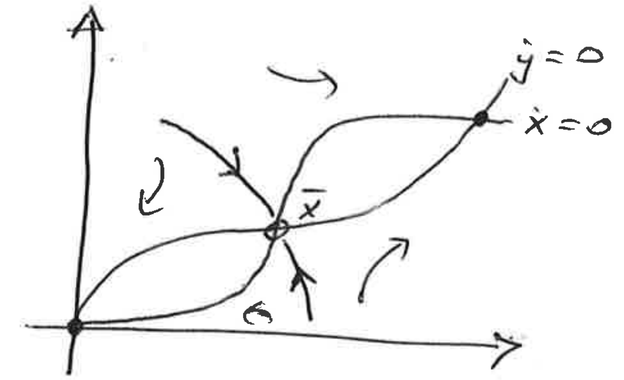


Fig: the stable manifold of $\bar{x}$ is the separatrix of the two basins of attraction.

(25%)

(d) Let $\dot{x} = f(x)$ and V a Lyapunov function:

$V(0) = 0$; $V(x) > 0$ for $x \neq 0$;

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot f(x) \leq -W(x) \leq 0$$

Analysis shows that solutions converge to the set where $W(x) = 0$. The invariance of limit sets further implies that solutions converge to the largest invariant set in that set.

(20%)

Example: damped pendulum

$$J\ddot{\theta} = -mgl \sin\theta - k\dot{\theta}$$

$x = (\theta, \dot{\theta}) = (0,0)$ is an equilibrium

and $V(x) = \frac{J\dot{\theta}^2}{2} + mgl\cos\theta$ is a Lyapunov function satisfying

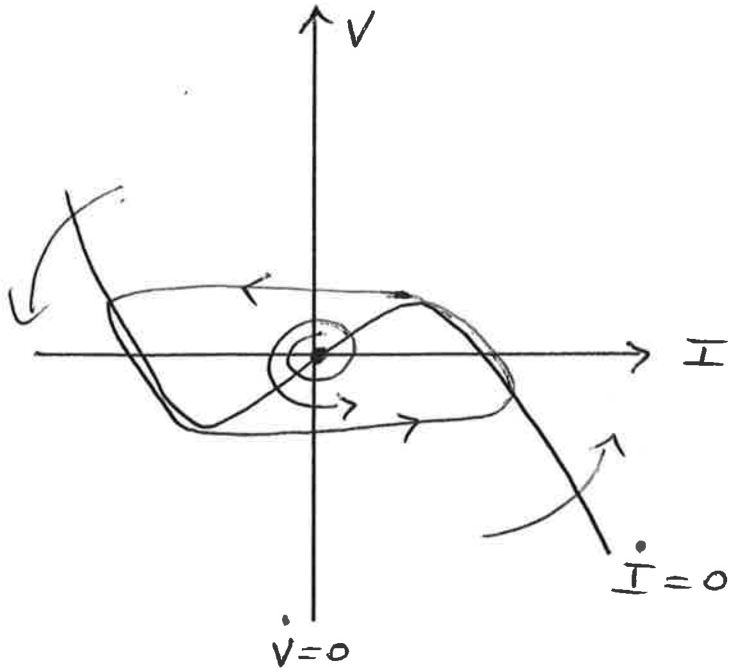
$$\dot{V} = -Jk\dot{\theta}^2 \leq 0$$

$\Rightarrow$ solutions converge to the set $E = \{x \mid \dot{\theta} = 0\}$

But only $\theta = 2k\pi$ and $\theta = k\pi$ are invariant sets in E. Therefore solutions converge to the downward or upward equilibrium. (15%)

4-

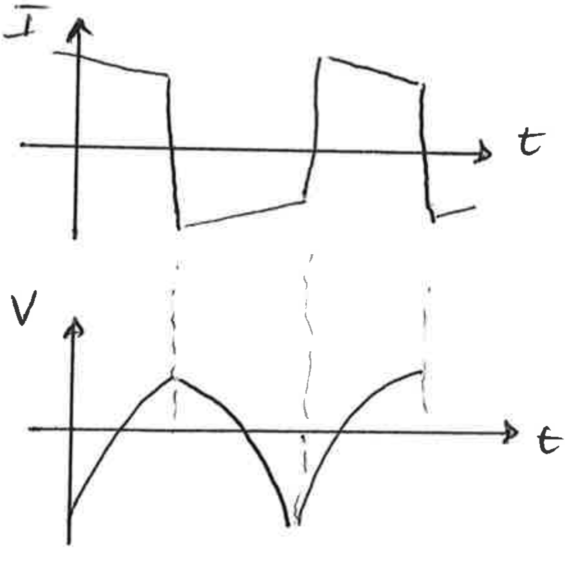
(a) (i)



Limit sets: One (unstable) equilibrium and one (stable) limit cycle.

$\frac{L}{C}$ small: trajectories are nearly horizontal away from the nullcline $\dot{I}=0$. (25%)

(ii)



The current is nearly constant while the capacitor slowly charges and gets almost instantaneously reverted, producing a square-wave oscillation. (20%)

b) (i) Write (1) as

LTI
$$\begin{cases} C\dfrac{dV}{dt} = I \\ L\dfrac{dI}{dt} = -V + I + u \\ y = I \end{cases}$$

NL
$$\begin{cases} u = \phi(y) \\ \phi(s) = s^3 \end{cases}$$

$H(s)$: $$LC\frac{d^2 I}{dt^2} = -I + C\frac{dI}{dt} + C\frac{du}{dt}$$

$$\Rightarrow \quad H(s) = \frac{Cs}{LCs^2 - Cs + 1}$$

(15%)

(ii) For an input $a = \cos\theta$, $\theta = \omega t$, the NL output is approximately

$$y(\theta) \simeq a_1 \cos\theta + \underbrace{b_1}_{0} \sin\theta$$

0 because odd NL

giving an equivalent gain

$$N(a) = \frac{a_1}{a}$$

For a cubic NL, one has

$$a_1 = \frac{4}{\pi}\int_0^{\pi/2} (a\cos\theta)^3 \cos\theta \, d\theta$$

$$= \frac{4a^3}{\pi}\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{3a^3}{4\pi} \qquad \Rightarrow \boxed{N(a) = \frac{3}{4\pi}a^2}$$

(20%)

(iii) $N(a)$ is real, therefore a limit cycle can exist if there exists a frequency for which $\operatorname{Im} H(j\omega) = 0$

But $H(j\omega) = \dfrac{C\sqrt{j\omega}}{1 - LC\omega^2 - C\sqrt{j\omega}}$

$\Rightarrow \quad \mathrm{Im}\, H(j\omega) = 0 \iff 1 - LC\,\omega^2 = 0$

$\iff \omega = \dfrac{1}{\sqrt{LC}}$

$\omega = \dfrac{1}{\sqrt{LC}}$ is the natural frequency of the resonant LC circuit.

(20%)

Q3 **Limits sets and Lyapunov analysis**

15 attempts, Average mark 11/20, Maximum 20, Minimum 3.

This question was primarily bookwork. Part (a) and (b) required a basic definition and the statement of an important theorem, which seems to have taken many candidates by surprise. Part (c) and (d) were done better. Many candidates lost marks because of a superficial understanding of concepts that require rigor.

Q4 **Van der Pol relaxation oscillator and describing function analysis**

16 attempts, Average mark 9/20, Maximum 16, Minimum 1.

The question with the lowest mean. Part (a): many students were unable to draw the required phase portrait.Many failed to draw the correct cubic nullcline and as a consequence did not find the limit cycle. Part (b) was better but many students struggled to express the equations as the feedback interconnection of a linear transfer function and a static nonlinearity (Part (b)(i)). In contrast, most students showed a correct understanding of the describing function method (Part (b)(ii)). The percentages of those two subquestions was changed from 20, 15 to 15, 20, respectively, in order to give more weight to the part of the question that was best answered.