Version RS/7

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 29 April 2014   2 to 3.30

Module 4F2

ROBUST AND NONLINEAR SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Let an uncertain multivariable system have transfer function

$$G(s) = (I + \Delta(s))G_o(s),$$

where $G_o(s)$ is the nominal plant transfer function and $\Delta(s)$ is a stable, unknown transfer function. Assume that $G_o(s)$ is internally stabilised by a controller $K(s)$ in the positive feedback convention. Derive a necessary and sufficient condition for the loop to be internally stable for all $\|\Delta(s)\|_\infty < \varepsilon$. You may assume the small gain theorem. [25%]

(b) A spinning body is to be controlled by two torque inputs applied about orthogonal axes. The available measurements are the angular velocities about the axes, subjected to a linear transformation. The transfer function relating the measurements to the torque inputs is given by

$$G_o(s) = \frac{1}{s^2 + a^2} \begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}$$

where a is a positive constant.

(i) Let $S(s) = (I + G_o(s))^{-1}$ and $T(s) = G_o(s)(I + G_o(s))^{-1}$. Show that

$$S(s) = \frac{1}{s+1} \begin{bmatrix} s & -a \\ a & s \end{bmatrix}$$

and find $T(s)$. [20%]

(ii) Assuming the positive feedback convention, deduce that the controller

$$K(s) = -I$$

internally stabilises $G_o(s)$. [20%]

(iii) Let $G_o(s)$ be subjected to a perturbation of the form described in part (a). Find the largest ε for which all such $G(s)$ are internally stabilised. [20%]

(iv) Describe tools that are needed to make the result of (b)(iii) less conservative if it is known that [15%]

$$\Delta(s) = \begin{bmatrix} \delta_1(s) & 0 \\ 0 & \delta_2(s) \end{bmatrix}.$$

2 Let $G(s)$ and $K(s)$ be the transfer function matrices of a plant and a controller which form an internally stable closed loop in a positive feedback configuration.

(a) Let $G = \tilde{M}^{-1}\tilde{N}$ be a normalised left coprime factorisation over H_∞.

(i) Explain what is meant for the factorisation to be left coprime and normalised. [15%]

(ii) Show that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} [I\ G] \right\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty . \qquad (1)$$

[15%]

(b) Briefly describe the H_∞ loop-shaping design procedure. [20%]

(c) Suppose that the norm in Equation (1) takes the value γ.

(i) Show that $\overline{\sigma}\left((I-GK)^{-1}\right)$ and $\overline{\sigma}\left(K(I-GK)^{-1}\right)$ are both bounded above by $\gamma\overline{\sigma}(\tilde{M})$ for any $s = j\omega$. [15%]

(ii) Show that $\overline{\sigma}\left(K(I-GK)^{-1}G\right)$ and $\overline{\sigma}\left((I-GK)^{-1}G\right)$ are both bounded above by $\gamma\overline{\sigma}(\tilde{N})$ for any $s = j\omega$. [15%]

(iii) Explain the consequences for the performance of the nominal feedback loop in the two cases $\gamma = 2.5$ and $\gamma = 100$. [20%]

[Hint: you may assume that $\overline{\sigma}(X) \leq \overline{\sigma}\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right)$ for matrices X and Y of compatible dimension.]

3 Consider the time-invariant differential equation $\dot{x} = f(x)$ in a n-dimensional vector space.

(a) Provide the definition of a limit set and state the three main properties of such sets. [15%]

(b) The Poincaré-Bendixson theorem restricts the possible limit sets of planar systems.

(i) Provide a statement of the theorem. [15%]

(ii) Draw an example of a limit set in the plane that is neither an equilibrium nor a limit cycle. Include in your drawing a sketch of a trajectory with this limit set. [15%]

(c) Define the stable manifold of a saddle point. Illustrate the importance of this object for the phase portrait of a bistable system. [20%]

(d) Explain how limit set properties can be exploited in the framework of Lyapunov theory. Illustrate your explanation with the asymptotic stability analysis of a physical model. [35%]

4 The Van der Pol nonlinear circuit is described by the following state-space model:

$$\begin{aligned} C\frac{d}{dt}V &= I \\ L\frac{d}{dt}I &= -V + I - I^3. \end{aligned}$$

The state variable V denotes the capacitor voltage and the state variable I denotes the inductor current.

(a) Consider the circuit when the ratio $\frac{L}{C}$ is small.

(i) Sketch the phase portrait of the circuit. Include in your drawing the current and voltage null-clines, the limit sets of the equation, and representative trajectories. [25%]

(ii) Explain why this circuit is called a relaxation oscillator and sketch the steady state behaviour of voltage and current as a function of time. [20%]

(b) (i) Show that the circuit can be represented as the feedback interconnection of a linear time-invariant system with transfer function $H(s)$ and a static nonlinearity NL and determine the expressions for $H(s)$ and NL. [15%]

(ii) Determine the describing function of the static nonlinearity. Note the identity $\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{3\pi}{16}$. [20%]

(iii) Use the describing function analysis method to predict the existence of a limit cycle and its frequency. Note that you are not required to determine the amplitude of the predicted limit cycle. Discuss the physical interpretation of the predicted frequency. [20%]

END OF PAPER

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Engineering Trips Part IIB
2014
Paper 4F2 : Robust and Nonlinear Systems and Control
Answers

1. (a) $\| (I - G_0 K)^{-1} G_0 K \|_\infty < \frac{1}{\epsilon}$
   (b) (i)

$$T(s) = \frac{1}{s+1} \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$$

   (iii) $\epsilon_{max} = \frac{1}{\sqrt{1+a^2}}$

2.

3. (b) (ii) For instance an eight-loop: $\omega(x)$ union of equilibrium and homoclinic orbits.
   (d) For instance a damped pendulum, Lyapunov function = energy, derivative is non negative but zero when velocity is zero.

4. (b) (i) $H(s) = \frac{C\,s}{LCs^2 - Cs + 1}$, $\phi(y) = y^3$.
   (ii) $\frac{3a^2}{4\pi}$
   (iii) $\omega = \frac{1}{\sqrt{LC}}$