~~Question 2~~ Q1)

(a)

$$\mathcal{F}_l\left(P(s), K(s)\right) = T_{\bar{w}\to\bar{z}}$$

$$\begin{aligned}
\bar{z} &= P_{11}\bar{w} + P_{12}\bar{u} \\
\bar{y} &= P_{21}\bar{w} + P_{22}\bar{u} \\
\bar{u} &= K\bar{y} \\
\bar{u} &= K(P_{21}\bar{w} + P_{22}\bar{u}) \\
\Rightarrow \bar{u} &= (I - KP_{22})^{-1}KP_{21}\bar{w} \\
\Rightarrow \bar{z} &= P_{11}\bar{w} + P_{12}\bar{u} = [P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}]\bar{w}
\end{aligned}$$

So

$$\mathcal{F}_l\left(P(s), K(s)\right) = T_{\bar{w}\to\bar{z}} = P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}$$

(b) Comparing with the formulation in the data sheet of the $\mathcal{H}_2$ optimal control problem we have

$$\begin{aligned}
&A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
&C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}
\end{aligned}$$

The matrices A_K, B_K, C_K in the state space realization of the controller are (from the data sheet),

$$\begin{aligned}
A_K &= A - B_2^T F - HC_2 \\
B_K &= -H \\
C_K &= F
\end{aligned}$$

where $F = B_2^T X$, $H = YC_2^T$. Substituting the expressions given for X, Y we have

$$\begin{aligned}
A_K &= \begin{bmatrix} 1-\alpha & 1 \\ -2\alpha & 1-\alpha \end{bmatrix}, \\
B_K &= -H = -\begin{bmatrix} 1 \\ 1 \end{bmatrix}\alpha, \\
C_K &= F = \begin{bmatrix} 1 & 1 \end{bmatrix}\alpha
\end{aligned}$$

(c) LQR problem. Comparing with the form in the data sheet, we have

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ Q = C_1^T C_1, \ R = 1, \ B = B_2, \ C = C_1$$

$$u_{opt} = -R^{-1}B^T X x$$

where X is the stabilising solution of the CARE in (b), and is hence as given in (b). Optimal cost

$$J_{opt} = x_0 X x_0 = 2\alpha$$

(d) In (b) we have an output feedback $\mathcal{H}_2$ optimal control problem, and the optimal controller involves an observer to estimate the state, together with a state feedback policy acting on the estimated state. The problem in (c), on the other hand, is a state feedback optimal control problem that can also be formulated as an $\mathcal{H}_2$ optimal control problem.

4F3 cribs

~~Question 1~~ Q2

(a) The value function is

$$V(x,k) = \min_{u_k,\dots,u_{h-1}} \left\{ x_h^2 + \sum_{i=k}^{h-1} (x_i^2 + u_i^2) \right\}$$

where $x_{k+1},\dots,x_h$ is the sequence generated with inputs $u_k,\dots,u_{h-1}$ given that $x_k = x$, i.e. it is the minimum remaining cost from step k onwards given that $x_k = x$. The dynamic programming equation is

$$\begin{aligned} V(x,k) &= \min_u \left\{ x^2 + u^2 + V(x_{k+1}, k+1) \right\} \\ &= \min_u \left\{ x^2 + u^2 + V(x+u, k+1) \right\} \end{aligned}$$

(b) Substituting $V(x,k) = g(k)x^2$ into the dynamic programming equation we get

$$g(k)x_k^2 = \min_u \left\{ x^2 + u^2 + g(k+1)(x+u)^2 \right\} \tag{1}$$

Differentiate w.r.t. u and set equal to 0 to find the minimizing value of u, i.e.

$$\begin{aligned} 2u + 2g(k+1)(x+u) &= 0 \\ \Rightarrow u &= -\frac{xg(k+1)}{1+g(k+1)} \end{aligned}$$

So

$$\begin{aligned} \min_u \{.\} &= x^2 + \left[\frac{xg(k+1)}{1+g(k+1)}\right]^2 + g(k+1)\left[x - \frac{xg(k+1)}{1+g(k+1)}\right]^2 \\ &= x^2 + x^2\frac{xg^2(k+1)}{(1+g(k+1))^2} + \frac{x^2 g(k+1)}{(1+g(k+1))^2} \\ &= x^2\left(1 + \frac{g(k+1)}{1+g(k+1)}\right) \end{aligned}$$

Hence from (1) we have

$$g(k) = 1 + \frac{g(k+1)}{1+g(k+1)}$$

(c) $h = 3$, $x_0 = 2$

$$
\begin{aligned}
g(h) &= 1 \text{ since } V(x_h, h) = x_h^2 \\
g(2) &= 1 + \frac{g(3)}{1+g(3)} = 1 + 1/2 = 3/2 \\
g(1) &= 1 + \frac{g(2)}{1+g(2)} = 1 + \frac{3/2}{1+3/2} = 1 + 3/5 = 8/5 \\
g(0) &= 1 + \frac{g(1)}{1+g(1)} = 1 + \frac{8/5}{1+8/5} = 1 + 8/13 = 21/13
\end{aligned}
$$

Minimum cost is $V(x_0, 0) = 2^2 \times 21/13 = 84/13$

(d)

$$
\tilde{J} = \tilde{x}_h^2 + \sum_{k=0}^{h-1} (\tilde{x}_k^2 + \tilde{u}_k^2)
$$

Also the system under the transformation becomes

$$
\tilde{x}_{k+1} = \alpha \tilde{x}_k + \alpha \tilde{u}_k
$$

From the data sheet, or from the previous expressions with $g(k+1)$ replaced with $\alpha^2 g(k+1)$, we have

$$
\tilde{u}_k = -(1 + \alpha^2 X_{k+1})^{-1} \alpha^2 X_{k+1} \tilde{x}_k
$$

where

$$
\begin{aligned}
X_{k-1} &= 1 + \alpha^2 X_k - \alpha^2 X_k (1 + \alpha^2 X_k)^{-1} \alpha^2 X_k \\
&= 1 + \alpha^2 X_k \left(1 - \frac{\alpha^2 X_k}{1 + \alpha^2 X_k} \right) \\
&= 1 + \frac{\alpha^2 X_k}{1 + \alpha^2 X_k}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\tilde{u}_k &= -\frac{\alpha^2 X_{k+1}}{1 + \alpha^2 X_{k+1}} \tilde{x}_k \\
\Leftrightarrow u_k &= -\frac{\alpha^2 X_{k+1}}{1 + \alpha^2 X_{k+1}} x_k
\end{aligned}
$$

3) a) Open loop optimal control problem solved over a finite horizon and starting from measurement of current state. First value of u used and then process repeated for next measured state.

b) $$J(x) = q x_0^2 + u_0^2 + q x_1^2 + u_1^2$$
$$= q x_0^2 + u_0^2 + q(2x_0 + u_0)^2 + u_1^2$$
$$= 5q x_0^2 + (1+q)u_0^2 + 4q x_0 u_0 + u_1^2$$

minimum clearly has $u_1 = 0$

$$\frac{\partial J}{\partial u} = 2(1+q)u_0 + 4q x_0$$
$$= 0 \quad \text{at} \quad u_0 = -\frac{2q}{1+q} x_0$$

Closed loop is then $x_{k+1} = \left(2 - \frac{2q}{1+q}\right) x_k = \frac{2}{1+q} x_k$

$\Rightarrow$ pole at $z = \frac{2}{1+q}$ & $|z| < 1$ iff $q > 1$

c) Let k be as defined, then can write
$$\sum_{l=0}^{\infty} q x(l)^2 + u(l)^2 = \sum_{l=0}^{n} q x(l)^2 + u(l)^2 + k\, x(n+1)^2$$
where the final term represents the optimal cost from $l = n+1$ onwards.

Now need to find a terminal set which is invariant and constraint admissible wrt the control law $u = -Lx$

$L > 1$, as control law is stabilising and so a suitable constraint admissible set is $|x| \leq \frac{1}{L}$. Is this invariant?

Yes, if $|x_k| \leq \frac{1}{L}$ then $x_{k+1} = 2x_k - Lx_k$
$|x_{k+1}| \leq (2-L) \cdot \frac{1}{L} = \frac{2}{L} - 1 = \frac{2-L}{L} \leq \frac{1}{L}$

Hence, receding horizon control law is to minimise
$$\sum_{k=0}^{n} \left(q x_k^2 + u_k^2\right) + k\, x_{n+1}^2 \quad \text{s.t.} \quad |x_{n+1}| \leq \frac{1}{L}$$
subject to $x_0 = x(k)$ and apply the control law $u(k) = u_0^*$.

4) a) $Q(x,a)$ is the optimal cost after taking action a at state x

$$Q(x,a) = r(x,a) + \min_{a'} Q(f(x,a), a')$$

b) It is required that the cost to be optimized is a sum of stage costs $r(x,a)$. For i) it is, with stage cost $u_k^2 + x_k^2$; for ii) it is not

c) $r(x,a) = |x| + |a|$

$\Rightarrow Q_1 =$

x \ u	-2	-1	0	1	2
4	6	5	4	X	X
3	5	4	3	4	X
2	4	3	2	3	4
1	X	2	1	2	3
0	X	X	0	1	2

$$Q_2(x,a) = |x| + |a| + \min_{a'} Q(x+a, a') =$$

x \ u	-2	-1	0	1	2
4	8	8	8	X	X
3	6	6	6	8	X
2	4	4	4	6	8
1	X	2	2	4	6
0	X	X	0	2	4

$Q_3 =$

x \ u	-2	-1	0	1	2
4	10	11	12	X	X
3	7	8	9	12	X
2	4	5	6	9	12
1	X	2	3	6	9
0	X	X	0	3	6

$Q_4 =$

x \ u	-2	-1	0	1	2
4	10	12	14	X	X
3	7	8	10	14	X
2	4	5	6	10	14
1	X	2	3	6	10
0	X	X	0	3	6

$Q_5 =$

x \ u	-2	-1	0	1	2
4	10	12	14	X	X
3	7	8	10	14	X
2	4	5	6	10	14
1	X	2	3	6	10
0	X	X	0	3	6

$= Q_4$

$\Rightarrow Q_4$ is optimal

Optimal strategy for $x_0 = 2$ is $u_0 = -2$, $\Rightarrow$ cost $= 4$

For $x^2 + u^2$ cost of this strategy is $\underline{\underline{16}}$

cost of $u_0 = -1$, $u_1 = -1$ is $4 + 1 + 1 + 1$
$= \underline{\underline{7}}$

which is hence a better strategy