Version ENH/5

EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 6 May 2014   2 to 3.30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F3 Optimal and Predictive Control data sheet (2 pages)
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the linear discrete-time plant with state $x(k)$, input $u(k)$ and controlled output $z(k)$ described by

$$x(k+1) = Ax(k) + Bu(k), \qquad z(k) = Cx(k).$$

(a) Define the state and input increments $\Delta x = x(k) - x(k-1)$ and $\Delta u = u(k) - u(k-1)$. If Δu (rather than u) is to be the manipulated variable determined by the control law, an equivalent system can be constructed of the form

$$\begin{bmatrix} \Delta x(k+1) \\ z(k+1) \end{bmatrix} = F \begin{bmatrix} \Delta x(k) \\ z(k) \end{bmatrix} + G\Delta u(k).$$

(i) Derive the values of F and G in terms of A, B and C. [20%]

(ii) Let Δu_i denote a prediction of Δu at time $k+i$, Δx_i denote a prediction of Δx at time $k+i$, and z_i denote a prediction of z at time $k+i$. Define

$$\Delta \underline{\mathbf{u}} = \begin{bmatrix} \Delta u_0^T & \Delta u_1^T & \cdots & \Delta u_{N-1}^T \end{bmatrix}^T.$$

A given predictive controller is designed to minimise the cost function

$$J(\Delta x(k), z(k), \Delta \underline{\mathbf{u}}) = \begin{bmatrix} \Delta x_N \\ z_N \end{bmatrix}^T P \begin{bmatrix} \Delta x_N \\ z_N \end{bmatrix} + \sum_{i=0}^{N-1} \left(\begin{bmatrix} \Delta x_i \\ z_i \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \Delta x_i \\ z_i \end{bmatrix} + \Delta u_i^T R \Delta u_i \right)$$

State sufficient conditions on P, Q_1, Q_2 and R that will guarantee that $J(\Delta x(k), z(k), \Delta \underline{\mathbf{u}})$ is a convex function of $\Delta \underline{\mathbf{u}}$ for a given $\Delta x(k)$ and $z(k)$. [15%]

(iii) Why is convexity of the cost function important? [10%]

(b) The controller from part (a) must be modified so that the output z tracks a non-zero, but constant reference $r(k) = r$.

(i) Assuming a perfect plant-model match and no additional disturbances, derive an equation that would be satisfied by compatible steady state values x_∞ and u_∞ (*not* their respective increments). Specify conditions in terms of A, B, and C under which a suitable pair (x_∞, u_∞) will exist for any r. When is this pair unique? [20%]

(ii) Let Δx_∞, z_∞ and Δu_∞ be steady-state values achieved by $\Delta x(k)$, $z(k)$ and $\Delta u(k)$ respectively when the reference is tracked with zero error. What should the values of Δx_∞, z_∞, and Δu_∞ be? [10%]

(iii) How should the cost function from part (a) be modified? [10%]

(iv) What are the advantages and disadvantages of using a plant model expressed in terms of Δu and Δx to design a constrained predictive controller? [15%]

2 Model predictive control usually employs the receding horizon principle.

(a) (i) Describe the receding horizon principle, in the context of predictive control. [20%]

(ii) What are the main advantages and disadvantages of receding horizon control compared with infinite horizon control? [10%]

(b) Consider the following linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

where $x(k)$ is the state, and $u(k)$ is the input. Let x_i and u_i be the predicted state and input at time $k+i$, i.e. $x_0 = x(k)$, and $x_{i+1} = Ax_i + Bu_i$, for $i = 0, 1, \ldots$. Define

$$\underline{\mathbf{x}} \triangleq \begin{bmatrix} x_0^T & x_1^T & x_2^T & x_3^T \end{bmatrix}^T, \quad \underline{\mathbf{u}} \triangleq \begin{bmatrix} u_0^T & u_1^T & u_2^T \end{bmatrix}^T$$

(i) Find matrices Φ and Γ such that [20%]

$$\underline{\mathbf{x}} = \Phi x(k) + \Gamma \underline{\mathbf{u}}$$

(ii) Suppose constraints are given in the form:

$$\begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} \leq e, \quad i = 0, 1, 2$$

$$T x_3 \leq t$$

Find matrices F and G and vector g such that these constraints can be written as [15%]

$$F\underline{\mathbf{x}} + G\underline{\mathbf{u}} \leq g.$$

(iii) Using the results of parts (i) and (ii) find matrices E and M and a vector h in terms of Φ, Γ, F, G and g such that the constraints in (ii) can be expressed as [10%]

$$E\underline{\mathbf{u}} \leq h + Mx(k).$$

(c) The final state x_N in the prediction horizon of length N, may be constrained to be in a subset of the state space that is admissible with respect to the constraint $Cx + Du \leq e$, and positively invariant under a given stabilising control law, $u = Kx$.

(i) What is meant by a positively invariant admissible set? [10%]

(ii) Briefly outline reasons for choosing such a terminal constraint in receding horizon control. [15%]

3 (a) For a discrete-time system satisfying the state equation

$$x_{k+1} = g(x_k, u_k)$$

with x_0 given, it is desired to minimise the cost function

$$J_d(x_0, u_0, u_1, \ldots, u_{h-1}) = \sum_{k=0}^{h-1} c_d(x_k, u_k) + J_h(x_h).$$

(i) Explain the concept of dynamic programming, and discuss its power and limitations. [20%]

(ii) Show that the corresponding dynamic programming equation for the "cost to go" or "value function" is: [20%]

$$V_d(x, k) = \min_{u_k} \left(c_d(x, u_k) + V_d(x_{k+1}, k+1) \right).$$

(b) A continuous-time system satisfies the state equation

$$\dot{x}(t) = f(x(t), u(t))$$

with $x(0)$ given. It is desired to determine an input function $u(\cdot)$ that minimises the cost function

$$J(x(\cdot), u(\cdot)) = \int_0^T c(x(\tau), u(\tau))\, \mathrm{d}\tau + J_T(x(T))$$

(i) Using the discrete-time dynamic programming equation from part (a) derive the continuous-time dynamic programming equation (i.e. the Hamilton-Jacobi-Bellman (HJB) Equation).
(Hint, let $V(x,t) = \min_u (c(x,u)\delta t + V(x + \delta x, t + \delta t))$ and let $\delta t \to 0$) [40%]

(ii) Let

$$f(x,u) = 2x + u, \quad c(x,u) = x^2 + u^2, \quad J_T(x) = 10x^2.$$

The corresponding value function has the form $p(t)x(t)^2$. Using the HJB equation, derive the differential equation that p satisfies, and state the boundary condition. [20%]

4 Consider the stable continuous-time plant

$$G(s) = C(sI - A)^{-1}B$$

(a) Define the $\mathcal{L}_2$ norm of a signal u as

$$\|u\|_2 = \sqrt{\int_0^\infty u(t)^T u(t)\, \mathrm{d}t}.$$

Considering a system $y = G(s)u$, state how the $\mathcal{H}_2$ norm of the plant $G(s)$, $\|G(s)\|_2$, relates $\|u\|_2$ to y. Provide definitions of any further signal norms used in your answer and explain their significance. [20%]

(b) The observability gramian is defined as

$$W_o(t_1) \triangleq \int_0^{t_1} e^{A^T\tau} C^T C e^{A\tau}\, \mathrm{d}\tau.$$

(i) Show that if A is stable, and $Q = \lim_{t_1 \to \infty} W_o(t_1)$, then Q satisfies the Lyapunov Equation: [25%]

$$A^T Q + QA + C^T C = 0.$$

(ii) Show that if A is stable, and $Q = Q^T > 0$ (note the strict inequality) then (A, C) is observable. [20%]

(iii) Write down an expression for the impulse response matrix $g(t)$ of the plant $G(s)$ as defined above. [5%]

(iv) Hence derive an expression for $\|G(s)\|_2$ in terms of Q. [30%]

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4F3 2014: Numerical answers

Question 1.

(a)(i) $F = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}$, $G = \begin{bmatrix} B \\ CB \end{bmatrix}$

(ii) $P \geq 0$, $Q_1 \geq 0$, $Q_2 \geq 0$, $R \geq 0$. ($R > 0$ for uniqueness)

(iii) Local = global optimum. Efficient algorithms.

(b)(i) $\begin{bmatrix} (A-I) & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$ or $\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$

Existence: Full row rank

Uniqueness: Invertible

(ii) $\Delta x_\infty = 0$, $\Delta u_\infty = 0$, $z_\infty = r$.

(iii) $z_i \leftarrow z_i - r$, $z_N \leftarrow z_N - r$.

(iv) –

Question 2.

(a)(i) –

(ii) –

(b)(i) $\Phi = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 & & \\ B & & \\ AB & B & \\ A^2B & AB & B \end{bmatrix}$ (ii) $F = \begin{bmatrix} C & & & \\ & C & & \\ & & C & \\ & & & T \end{bmatrix}$, $G = \begin{bmatrix} D & & \\ 0 & D & \\ 0 & 0 & D \\ 0 & 0 & 0 \end{bmatrix}$, $g = \begin{bmatrix} e \\ e \\ e \\ t \end{bmatrix}$

(iii) $(F\Gamma + G)\underline{\mathbf{u}} \leq g - F\Phi x(k)$

(c)(i)

$$\mathbb{T} = \left\{ x : (C + DK)(A + BK)^i x \leq e, \quad \forall i = 0, \ldots, \infty \right\}.$$

(ii) Recursive feasibility of RHC in closed loop

Question 3.

(a) –

(b)(i) –

(ii) $-\dot{p}(t) = 1 - p(t)^2 + 4p(t)$, $P(T) = 10$.

Question 4.

(a) $\|y\|_\infty \leq \frac{1}{2\pi}\|G(s)\|_2\|u\|_2$., where $\|y\|_\infty = \sup_t \sqrt{y(t)^T y(t)}$

(b)(i) –

(ii) –

(iii) $g(t) = Ce^{At}B$

(iv) $\|G(s)\|_2 = \sqrt{2\pi}\sqrt{\text{trace}(B^T Q B)}$

E. N. Hartley (Principal Assessor)
12 May 2014.