4F3: Optimal and Predictive Control Numerical Answers (2015)
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1. (a) $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, $\exists u = \kappa(x)$ such that $V(Ax + Bu) \leq V(x)$.

   (b)(i)

$$\begin{aligned} V(x_1^*) &\leq J(x_1^*, \tilde{\mathbf{u}}) \\ J(x_1^*, \tilde{\mathbf{u}}) &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 - x_N^{*T} P x_N^* + x_N^{*T} Q x_N^* + x_N^{*T} A^T P A x_N^* \\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 + \underbrace{x_N^{*T}(A^T P A - P + Q) x_N^*}_{0} \\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 \leq V(x_0) \end{aligned}$$

   (ii) $(A + BK)^T P (A + BK) - P = -Q - K^T R K$

   (c) $\sum_{i=N_1}^{\infty} x_i^T Q x_i = \sum_{i=N_1}^{\infty} \left(x_i^T P x_i - x_i A^T P A x_i \right) = \lim_{i \to \infty} x_{N_1}^T P x_{N_1} + x_i^T P x_i = x_{N_1}^T P x_{N_1}$ since $\rho(A) < 1$

2. (a) (i) – (ii) –

   (b)

$$H = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \quad h = \begin{bmatrix} -Ru_s \\ -Qx_s \\ -Ru_s \\ -Px_s \end{bmatrix}, \quad F = \begin{bmatrix} B & -I & 0 & 0 \\ 0 & A & B & -I \end{bmatrix}, \quad f = \begin{bmatrix} -Ax(k) - d \\ -d \end{bmatrix}$$

$$G = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & M & E & 0 \\ 0 & 0 & 0 & M_N \end{bmatrix}, \quad g = \begin{bmatrix} -Mx(k) + b \\ b \\ b_N \end{bmatrix}$$

   (c)

$$\underline{\theta} = \begin{bmatrix} x_s \\ u_s \\ \hat{r} \end{bmatrix}, \quad F = \begin{bmatrix} (A - I) & B & 0 \\ C & D & -I \end{bmatrix}, \quad f = \begin{bmatrix} -d \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} M & E & 0 \\ M_N & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} b \\ b_N \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & & \\ & 0 & \\ & & I \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}.$$

3. (a)(i) $\|y\|_2 \leq \|G\|_\infty \|w\|_2$, $\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} y(t)^T y(t)\, \mathrm{d}t}$ (energy in signal)

   (ii) $\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$ (largest magnitude of the gain of the system over all frequencies)

   (b) $u = -B_2^T X x = -\begin{bmatrix} \alpha^{-1/2} & \sqrt{2}\alpha^{-3/4} \end{bmatrix} x = -\begin{bmatrix} (1 - \gamma^{-2})^{-1/2} & \sqrt{2}(1 - \gamma^{-2})^{-3/4} \end{bmatrix} x.$

   (c) Use interval bisection.

   (d) $\begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies$ not full rank so not observable.

4. (a)(i) $f((1 - \alpha)x + \alpha y) \leq (1 - \alpha) f(x) + \alpha f(y)$, for $\alpha \in [0, 1]$

   [or equivalently, $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$],

   (ii) Two points in a shape connected by straight line exiting shape.

   (iii) Local = global minimum. Efficient solution.

   (b) (i) $X_2 \geq 0$, $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0$.

   (ii) $u_1 = -\begin{bmatrix} 1 & 7/5 \end{bmatrix} x_1$.

   (iii) $x^T X_2 x$, 134.

   (c) DLQR $\implies$ apply whole input sequence, MPC $\implies$ receding horizon: recompute sequence at each time step.