Version RV/4

EGT3
ENGINEERING TRIPOS PART IIB

Monday 28 April 2014   9.30 to 11.00

Module 4F5

ADVANCED COMMUNICATIONS AND CODING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 4F5 Advanced Communications and Coding data sheet (5 pages).
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Let X be a discrete random variable that takes values in $\{-1, 1\}$ with

$$Pr(X=-1) = Pr(X=1) = \frac{1}{2}.$$

Let Y be a continuous random variable with uniform distribution in the interval $[-2, 2]$. Assume that Y is independent of X. Define a third random variable Z as

$$Z = X + Y.$$

(i) Obtain and sketch the probability density function (pdf) of Z. [25%]

(ii) Calculate the differential entropy of Z. [15%]

(b) For any two discrete random variables X, Y with joint probability mass function (pmf) P_{XY}, show that

$$H(X \mid X+Y) = H(Y \mid X+Y).$$

Hint: It may be useful to consider $H(X, Y \mid X+Y)$. [25%]

(c) Show that the capacity of any discrete memoryless channel is less than or equal to

$$\min\{\log_2 M,\ \log_2 N\} \ \text{ bits},$$

where M is the size of the input alphabet and N is the size of the output alphabet. [35%]

2 Consider arithmetic over the Galois Field GF(7).

Note that for length-3 vectors in GF(7), the Discrete Fourier Transform (DFT) and its inverse over GF(7) can be computed by multiplication with the matrices

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{F}^{-1} = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 6 & 3 \\ 5 & 3 & 6 \end{bmatrix},$$

respectively.

(a) Find an element β of multiplicative order 6 in GF(7). What is the value and the multiplicative order of $\alpha = \beta^2$? [10%]

(b) Specify a 2×3 parity-check matrix for the Reed Solomon code of length 3 over GF(7). What is the code dimension? Is the parity-check matrix unique? [20%]

(c) Specify an encoding matrix for the code. How many codewords does the code contain? [15%]

(d) A codeword has been transmitted over a noisy channel and the corresponding received vector is $\mathbf{r} = [2, 0, 1]$. Assume that the channel causes at most one error over a block of 3 code symbols.

(i) Define the *linear complexity* of a finite length sequence. [10%]

(ii) What is the linear complexity of the DFT of the error sequence? Justify your answer. [15%]

(iii) Find the shortest Linear Feedback Shift Register (LFSR) that generates the DFT of the error sequence. [10%]

(iv) Use the LFSR and the inverse DFT to recover the error sequence in the time domain. [10%]

(v) Determine the transmitted codeword. [10%]

3 (a) Consider a binary input, binary output discrete memoryless channel where the input $X \in \{0,1\}$ gets multiplied by an independent noise variable $Z \in \{0,1\}$ to produce the output Y. That is,

$$Y = XZ.$$

Let $Pr(Z=1) = a$ and $Pr(Z=0) = (1-a)$.

(i) Find the capacity of this channel and the maximising input distribution on X. Your answers should be in terms of the parameter a. [40%]

Hint: Note that

$$\frac{d}{dp} H_2(ap) = a \log_2 \left(\frac{1-ap}{ap} \right),$$

where $H_2(p)$ is the binary entropy function and a is any non-zero constant.

(ii) Describe briefly how you would construct a capacity achieving code for this channel when $a = 1/2$. (Assume you don't have to worry about the complexities of encoding, decoding, and storage.) [25%]

(b) A binary linear code is specified by the following parity-check matrix

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(i) Draw the factor graph corresponding to the parity-check matrix. [15%]

(ii) A codeword from this code is transmitted over a binary erasure channel (BEC) and the received sequence is $[1, 0, \varepsilon, \varepsilon, 0, 0, 1, \varepsilon]$, where ε denotes an erased output. What was the transmitted codeword? Explain how you recovered it. [20%]

4 Consider the constellation consisting of three symbols $\{-A, 0, A\}$, shown in Fig. 1.

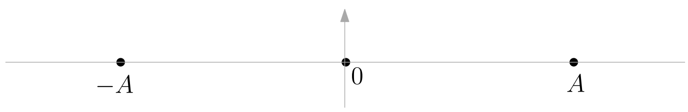

Fig. 1

(a) Suppose that this constellation is used to signal over the discrete-time Additive White Gaussian Noise (AWGN) channel

$$Y = X + N$$

where the noise N is distributed as $\mathcal{N}(0, \frac{N_0}{2})$, i.e., it is Gaussian with zero mean and variance $\frac{N_0}{2}$. Assume that all the constellation symbols are equally likely.

(i) Find the average symbol energy E_s of the constellation. [10%]

(ii) Write down the decision regions for the optimal detector. [15%]

(iii) Compute the probability of detection error and express it in terms of the ratio $\frac{E_s}{N_0}$. (Your answer should involve the Q-function.) [25%]

(b) Now suppose that the constellation in Fig. 1 is used to signal over the fading channel

$$Y = hX + N$$

where $h \sim \mathcal{CN}(0,1)$ and $N \sim \mathcal{CN}(0, N_0)$ are complex Gaussian random variables. Assume that coherent detection is performed, i.e., the fading coefficient h is known at the receiver.

(i) At the receiver, we project Y in the direction of h by multiplying it by $\frac{h^*}{|h|}$, and then perform detection. (h^* is the complex conjugate of h.) What is the probability of detection error conditioned on h? [15%]

(ii) Use the approximation $Q(x) \approx \frac{1}{2}e^{-x^2/2}$ for the Q-function, and compute the probability of detection error averaged over all realisations of h. Express your answer in terms of the ratio $\frac{E_s}{N_0}$. Note that the probability density function of $|h|^2$ is given by $f_{|h|^2}(x) = e^{-x}, \;\; x \geq 0$. [15%]

(iii) Compare the average probability of error for the fading channel with the probability of error for the AWGN channel in part (a). Briefly describe one technique for improving the probability of error for the fading channel. [20%]

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Module 4F5: Engineering Tripos 2013/14 – Numerical Answers

1. a) $f_Z(z) = 1/4$ for $z \in [-1,1]$, and $1/8$ for $z \in [-3,-1) \cup (1,3]$

2. a) $\beta = 3$, $\alpha = 2$ b) $\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ c) $\mathbf{G}_{\text{sys}} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$, 7 codewords
   d) ii) Linear complexity=1 iv) $[0, 3, 0]$ v) $[2, 4, 1]$

3. a) $p^* = [a(1 + 2^{H_2(a)/a})]^{-1}$, $\mathcal{C} = \log_2(1 + 2^{H_2(a)/a}) - \frac{H_2(a)}{a}$ b) ii) $[1, 0, 1, 1, 0, 0, 1, 0]$

4. a) i) $2A^2/3$ iii) $\frac{4}{3}\mathcal{Q}(\sqrt{3E_s/4N_0})$
   b) i) $\frac{4}{3}\mathcal{Q}(\sqrt{3|h|^2 E_s/4N_0})$ ii) $\frac{2}{3}\left(1 + \frac{3}{8}\frac{E_s}{N_0}\right)^{-1}$