Version SJG/4

EGT3
ENGINEERING TRIPOS PART IIB

Wednesday 30 April 2025 9.30 to 11.10

Module 4F7

STATISTICAL SIGNAL AND NETWORK MODELS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) The Kalman filtering recursions for a linear state-space model, based on data y_1 , y_2, ...y_t, are expressed as

$$\begin{aligned}
\mu_{t|t-1} &= A\mu_{t-1|t-1} \\
P_{t|t-1} &= \Sigma_v + AP_{t-1|t-1}A^T \\
\mu_{t|t} &= \mu_{t|t-1} + K_t(y_t - B\mu_{t|t-1}) \\
P_{t|t} &= (I - K_t B)P_{t|t-1} \\
K_t &= P_{t|t-1}B^T(P_{t|t-1}B^T + \Sigma_w)^{-1}
\end{aligned}$$

where the terms $\mu_{t|t-1}$ and $\mu_{t|t}$ are estimators of a hidden state variable x_t.

(i) Write down an appropriate linear state-space model that is solved by these recursions, including the initialisation at $t = 0$. What do the terms Σ_v and Σ_w signify? [10%]

(ii) Under what conditions on the state-space model do $\mu_{t|t-1}$ and $\mu_{t|t}$ give optimal Bayesian estimators of the state? [10%]

(iii) Under these conditions, what are the conditional distributions $p(x_t|y_{1:t})$ and $p(x_t|y_{1:t-1})$? [15%]

(iv) When these conditions may not be assumed, what form of optimality is achieved by the Kalman filter recursions? [10%]

(b) A stochastic differential equation is defined as

$$\dot{x}(t) = ax(t) + W(t)$$

where $\{W(t)\}$ is continuous-time zero-mean Gaussian white noise with autocorrelation function $R_{WW}(\tau) = \delta(\tau)$ and $a \leq 0$.

(i) Using Laplace transforms, or otherwise, show that

$$x(\delta t) = \exp(a\delta t)\left(x(0) + \int_0^{\delta t} \exp(-a\tau)W(\tau)d\tau\right)$$

[20%]

(ii) Determine the mean and variance of the integral term $\int_0^{\delta t} \exp(-a\tau)W(\tau)d\tau$. [25%]

(iii) Hence write down a linear Gaussian dynamical model in discrete time for samples $x_n = x(n\delta t)$. [10%]

2 A nonlinear dynamical system obeys the following dynamical model for $t = 1, 2, ...,$:

$$x_t = \pi/2 \sin(x_{t-1}) + v_t$$

and the state is observed in noise:

$$y_t = x_t^2 + w_t$$

where $v_t \sim \mathcal{N}(0, \sigma_v^2)$ and $w_t \sim \mathcal{N}(0, \sigma_w^2)$ are mutually uncorrelated white noise processes.

(a) Derive a linear and Gaussian approximation of the dynamical system expanded around a fixed point $x_{t-1} = x^*$. [15%]

(b) Derive a linear and Gaussian approximation of the observation model, expanded around a fixed point $x_t = x^+$. [10%]

(c) Suppose that $p(x_{t-1}|y_{1:t-1}) = \mathcal{N}(\mu_{t-1}, p_{t-1}^2)$ where p_{t-1} is a standard deviation. Show that the prediction step under the approximate dynamical system is:

$$p(x_t|y_{1:t-1}) \approx \mathcal{N}(\pi/2 \sin(x^*) + (\mu_{t-1} - x^*)\pi/2 \cos(x^*),\ \pi^2/4 \cos^2(x^*) p_{t-1}^2 + \sigma_v^2)$$

[20%]

(d) Explain how sequential estimation of x_t in this approximate model can be implemented as an Extended Kalman Filter, stating carefully how x^* and x^+ are chosen and detailing how the update step of the Kalman Filter is modified compared with the standard KF. [35%]

(e) If this model were also implemented using a particle filter, using the 'prior' proposal distribution $p(x_t|x_{t-1})$, compare and contrast the likely performance of the two filters. Determine the incremental weight of the particle filter with this proposal distribution. [20%]

3 Consider a model for *directed* graphs on n nodes, where the edge from i to j occurs with probability p_{ij}. The adjacency matrix is defined such that

$$A_{ij} \sim \text{Bernoulli}(p_{ij})$$

and we allow self-edges to occur.

(a) Show that the log-likelihood for p_{ij} given adjacency matrix A is

$$\sum_{i,j} \left(A_{ij} \log p_{ij} + (1 - A_{ij}) \log\left(1 - p_{ij}\right)\right)$$

[20%]

(b) Find the maximum likelihood estimate (MLE) for the parameters p_{ij}. [20%]

(c) We now constrain the parameter matrix p_{ij} to be of the form

$$p_{ij} = \theta_i$$

Show that the MLE is

$$\hat{\theta}_i = \frac{1}{n} \sum_j A_{ij} = \frac{k_i}{n}$$

where $k_i = \sum_j A_{ij}$ is the out-degree of node i. [20%]

(d) A *reciprocal* edge is an edge that exists in both directions, i.e., $A_{ij} = A_{ji} = 1$ (note, we will not count self-edges as reciprocal). Show that the expected number of reciprocal edges at node i is

$$\theta_i \left(\sum_j \theta_j - \theta_i \right)$$

and the total number is thus

$$\left(\sum_i \theta_i \right)^2 - \sum_i \theta_i^2$$

[20%]

(e) For online social networks, both the number of edges and the number of reciprocal edges typically scale linearly with n. Show that in this model, one cannot have both. [20%]

4 Consider a configuration model network that contains only nodes of degree 1 and degree 3. A fraction α of the nodes have degree 1, while a fraction $1 - \alpha$ have degree 3. Hence, the degree distribution is

$$p_k = \alpha\, \delta_{k,1} + (1 - \alpha)\, \delta_{k,3}$$

(a) What is the mean degree? [10%]

(b) Provide an expression for the excess degree distribution, q_k. [15%]

(c) Let u be the probability that a randomly chosen edge does not lead to the giant component. Show that u obeys the self-consistent equation

$$\left(u - 1\right)\left(u - \frac{\alpha}{3 - 3\alpha}\right) = 0$$

[35%]

(d) Hence, argue that a giant component exists if and only if $\alpha < 3/4$. [15%]

(e) Derive an expression for the fractional size of the giant component when $\alpha = 3/5$. [25%]

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