Question 1

(a) State the algorithm (draw a block diagram), state the cost function being minimised, state how μ should be chosen (bound for stability, trade-off between tracking performance and steady state mean square error), state limit of $E\{\underline{h}(n)\}$. Could also mention the variants of LMS.

(b) (i) modified LMS:

$$\underline{h}(n+1) = \underline{h}(n) + (\mu R^{-1})\,\underline{u}(n)\,e(n)$$

$$e(n) = d(n) - \underline{u}(n)^T \underline{h}(n)$$

Invoking independence assumption:

$$E\{\underline{h}(n+1)\} \approx E\{\underline{h}(n)\} + \mu R^{-1}\left(\underline{p} - R\,E\{\underline{h}(n)\}\right)$$

$$= E\{\underline{h}(n)\} + \mu\left(\underline{h}_{opt} - E\{\underline{h}(n)\}\right)$$

$$\left(E\{\underline{h}(n+1)\} - \underline{h}_{opt}\right) = (1-\mu)\left(E\{\underline{h}(n)\} - \underline{h}_{opt}\right)$$

Thus

$$E\{\underline{h}(n+1)\} \to \underline{h}_{opt}$$

provided $|1-\mu| < 1$

(ii) $\mu = 1$ from analysis in part (i)

(iii) Could use sample mean estimate of $\underline{R} = E\{\underline{u}(n)\,\underline{u}(n)^T\}$

or the RLS's exponential estimate

$$\underline{R}(n) = \lambda\,\underline{R}(n-1) + \underline{u}(n)\,\underline{u}(n)^T$$

with $0 < \lambda < 1$.

Could use the online calculation of $\underline{R}(n)$ as in the RLS algorithm

(iv) LMS sensitive to eigenvalue spread of $\underline{R}$. Cost $O(M)$

RLS insensitive to eigenvalue spread but costs $O(M^2)$

Modified LMS now performs like RLS.

(c) (i) $\alpha(n+1) = \alpha(n) + \mu\, x(n-1)\, e(n)$

$$e(n) = x(n) - \alpha(n)\, x(n-1)$$

where $\{\alpha(n)\}$ are estimates of α

(ii) $$\mu < \frac{2}{\lambda_{max}(R)}$$

$R = E\{x(n)^2\}$ (a scalar in this example)

$$E\{x(n)^2\} = \alpha^2 E\{x(n-1)^2\} + E\{w(n)^2\}$$

If stationary then $E\{x(n)^2\} = \dfrac{\sigma^2}{1-\alpha^2}$

Question 2

a (i) Quadratic cost function. Differentiate and set to 0 to get

$$\sum_{k=0}^{n} \lambda^{n-k} y(k) \underline{u}(k) = \left(\sum_{k=0}^{n} \lambda^{n-k} \underline{u}(k) \underline{u}(k)^T \right) \underline{h}$$

Multiply LHS and RHS by $(\dots)^{-1}$.

(ii)

$$R(n) = \sum_{k=0}^{n} \lambda^{n-k} \underline{u}(k) \underline{u}(k)^T$$

$$p(n) = \sum_{k=0}^{n} \lambda^{n-k} y(k) \underline{u}(k)$$

(iii)

$$R(n) = \lambda R(n-1) + u(n)^2$$

$$p(n) = \lambda p(n-1) + y(n) u(n)$$

$$h(n) = R^{-1}(n) p(n)$$

$$= \frac{\lambda p(n-1)}{\lambda R(n-1) + u(n)^2} + \frac{y(n) u(n)}{\lambda R(n-1) + u(n)^2}$$

$$= \frac{\lambda R(n-1)}{R(n)} h(n-1) + \frac{y(n) u(n)}{R(n)}$$

$$= h(n-1) + \left(\frac{\lambda R(n-1)}{R(n)} - 1 \right) h(n-1) + \frac{y(n) u(n)}{R(n)}$$

$$= h(n-1) + \left(\frac{-u(n)^2}{R(n)} \right) h(n-1) + \frac{y(n) u(n)}{R(n)}$$

$$= h(n-1) + \frac{u(n)}{R(n)} \left(y(n) - u(n) h(n-1) \right)$$

b(i) cost function is

$$\sum_{k=0}^{n} \lambda^{n-k} \left(x(k) - \beta\, x(k-1) \right)^2$$

From part (a) solution is

$$\beta = R(n)^{-1} p(n)$$

$$= \frac{\sum_{k=0}^{n} \lambda^{n-k} x(k)\, x(k-1)}{\sum_{k=0}^{n} \lambda^{n-k} x(k-1)^2}$$

(ii) For $\lambda = 1$, assuming ergodicity

$$\lim_{n \to \infty} R(n)^{-1} p(n) = \frac{E\{x(k)\, x(k-1)\}}{E\{x(k)^2\}}$$

(iii)

$$x(n) = \beta\, x(n-1) + w(n)$$

$$E\{x(n)\, x(n-1)\} = \beta\, E\{x(n-1)^2\} + \cancel{E\{x(n-1) w(n)\}}_{=0}$$

Thus limit of ratio is β.

Question 3

(a)
$$p(x_0, \ldots, x_{N-1} \mid a)$$
$$= p(x_0|a)\, p(x_1|x_0, a) \cdots p(x_{N-1}|x_0, \ldots, x_{N-2}, a) \quad \text{(prob. chain rule)}$$
$$= p(x_0|a) \cdots p(x_{N-1}|a) \quad \text{(by independence of } e_n\text{)}$$
$$= \prod_{k=0}^{N-1} \frac{1}{2\pi\sigma^2} \exp\left(\frac{|x_k - a\exp(i\omega_0 k)|^2}{2\sigma^2} \right)$$

Note $x_k = \mathrm{Re}\{a\exp(i\omega_0 k)\} + e_{k,R} + j\left(\mathrm{Im}\{a\exp(i\omega_0 k)\} + e_{k,I}\right)$

where $e_k = e_{k,R} + j\, e_{k,I}$

and $e_{k,R}$ is Gaussian $(0, \sigma^2)$ as is $e_{k,I}$. Also $e_{k,R}$ and $e_{k,I}$ are independent.

Thus $p(x_k|a) = p(\mathrm{Re}\{x_k\}|a) \times p(\mathrm{Im}\{x_k\}|a)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2} \mathrm{Re}\{x_k - a\exp(i\omega_0 k)\}^2 \right)$$
$$\times \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2} \mathrm{Im}\{x_k - a\exp(i\omega_0 k)\}^2 \right)$$

(b) Differentiate the log of the density wrt a:

$$\frac{d}{da} \sum_{k=0}^{N-1} |x_k - a\exp(i w_0 k)|^2 = 0$$

(ignoring terms not dependent on a)

(c)

$$|x_k - a\exp(i w_0 k)|^2$$
$$= |x_{R,k} - a\cos(w_0 k) + i(x_{I,k} - a\sin(w_0 k))|^2$$

$$\frac{d}{da}|\cdots|^2 = 2(x_{R,k} - a\cos w_0 k)(-\cos w_0 k) + 2(x_{I,k} - a\sin w_0 k)(-\sin w_0 k)$$

Thus

$$\sum_{k=0}^{N-1} (x_{R,k} - a\cos w_0 k)(\cos w_0 k) + (x_{I,k} - a\sin w_0 k)(\sin w_0 k) = 0$$

$$a\sum_{k=0}^{N-1} \cos^2 w_0 k + \sin^2 w_0 k = \sum_{k=0}^{N-1} \underbrace{x_{R,k}\cos w_0 k + x_{I,k}\sin w_0 k}_{\mathrm{Re}\{x_k \exp(i w_0 k)\}}$$

(c+d)

$$x_{n+k} x_n^* = \left(a\exp(i\omega_0(n+k)) + e_{n+k}\right)\left(a\exp(-i\omega_0 n) + e_n^*\right)$$

$$= a^2 \exp(i\omega_0 k) + e_{n+k} e_n^* + \underbrace{\text{cross terms}}_{\text{has zero expectation}}$$

$$E\{e_{n+k} e_n^*\} = E\{(e_{R,n+k} + i e_{I,n+k}) \times (e_{R,n} - i e_{I,n})\}$$

$$= E\{e_{R,n+k} e_{R,n}\} + E\{e_{I,n+k} e_{I,n}\}$$

$$= \begin{cases} 2\sigma^2 & k = n \\ 0 & \text{otherwise} \end{cases}$$

$$\text{DTFT}\{R_{xx}[k]\} = 2\pi a^2 \delta(\omega - \omega_0) + 2\sigma^2$$

e) The periodogram estimate of the power spectrum will be centered at ω_0. (See lecture notes on Expected value of the periodogram and give details.)

One could use the various enhancements of the periodogram estimate to extract this main peak's centre reliably, i.e. reducing variance of the periodogram.

Question 4

a) Give a description of the ARMA model, its power spectrum and the effect of adjusting the model order and coefficient values on its spectrum.

State the main methods for estimating the spectrum of the AR, MA and ARMA models.

Give some indication of how the variance of the Spectrum estimate behaves as a function of the data size.

b) (i) Starts at P to avoid using data for negative time since not given.

b) (ii) AR(P) model:

$$x_n = \sum_{k=1}^{P} a_k x_{n-k} + w_n$$

w_n = white noise, zero-mean, variance σ^2.

Prediction error:

$$x_n - \sum_{k=1}^{P} a_k x_{n-k}$$

$$= x_n - \underline{a}^T \underline{x}_{n-1}$$

where $\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_P \end{bmatrix}$, $\underline{x}_{n-1} = \begin{bmatrix} x_{n-1} \\ \vdots \\ x_{n-P} \end{bmatrix}$

differentiate to get

$$\sum_{n=p}^{N-1} 2\left(x_n - \underline{a}^T \underline{x}_{n-1}\right) \times \left(-\underline{x}_{n-1}\right)$$

set to zero, re-arrange gives answer

(iii) Divide LHS and RHS by $\frac{1}{N-P}$, take the expectation and we get the Yule-Walker equations. So this is the empirical version of YW equations.

(iv) multiply LHS and RHS of AR(P) model by x_n, take expectation to get

$$R_{xx}[0] = \sum_{k=1}^{P} a_k R_{xx}[k] + \sigma^2$$

Now use estimated $\underline{a}$, $R_{xx}[0], \ldots, R_{xx}[P]$ and solve for σ^2. (Again empirical YW solution.)

c(i) $$R_{xx}[k] = \begin{cases} 2 & k=0 \\ 0 & k=1 \\ 1 & k=2 \\ 0 & \text{otherwise} \end{cases}$$

Spectrum $= 2 + e^{-jw2} + e^{jw2}$

Question 4

(c)(ii) $\hat{S}_x(e^{j\omega T}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x_n e^{-j\omega T n}\right|^2$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x_n x_m e^{-j\omega n} e^{j\omega m}$$

for $|n-m| = 0$, $E\{x_n x_m\} = 2$

$|n-m| = 2$, $E\{x_n x_m\} = 1$

0 otherwise

$$= \frac{1}{N}\Big[2N + x_0 x_2 e^{j\omega 2}$$
$$+ x_1 x_3 e^{j\omega 2}$$
$$+ x_{N-2} x_{N-4} e^{-j\omega 2}$$
$$+ x_{N-1} x_{N-3} e^{-j\omega 2}$$
$$+ \sum_{n=2}^{N-3} \cancel{x_n(x_{n-2}+x_{n+2})}\; x_n x_{n-2} e^{-2\omega j} + x_n x_{n+2} e^{2\omega j}$$
$$+ \dots \Big]$$

these unlisted terms have zero expectation $\longrightarrow + \dots$

$$E\{\dots\} = 2 + \frac{2}{N}\left(e^{j\omega 2} + e^{-j\omega 2}\right) + \left(e^{2\omega j} + e^{-2\omega j}\right)\frac{(N-4)}{N}$$

True spectrum $= 2 + e^{-j\omega 2} + e^{j\omega 2}$

and Periodogram is asymptotically unbiased.