EGT3
ENGINEERING TRIPOS PART IIB

Friday 2 May 2014 9.30 to 11

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Describe in detail the Least Mean Square (LMS) algorithm for adaptive filtering, with all signals clearly defined. [20%]

(b) The constant step-size μ of the LMS algorithm is replaced by $\mu\mathbf{R}^{-1}$ where $\mathbf{R}$ is the input signal autocorrelation matrix.

(i) Determine the conditions on μ for the expected value of the filter coefficients of this modified LMS algorithm to converge to the Wiener solution. [30%]

(ii) State, with reason, the value of μ for which convergence is the fastest. [5%]

(iii) Since $\mathbf{R}$ is not known in general, propose a practical implementation of this modified LMS algorithm. [10%]

(iv) Contrast the performance of this modified LMS algorithm with that of the LMS and exponentially weighted Recursive Least Squares algorithm. [10%]

(c) You are given samples $x(n)$ from the AR(1) model

$$x(n) = \alpha x(n-1) + w(n)$$

where $|\alpha| < 1$ and $w(n)$ are independent and identically distributed zero-mean random variables with variance σ^2.

(i) Describe an implementation of the LMS algorithm to learn α. [10%]

(ii) Give a bound on the LMS step-size, in terms of the parameters of this AR(1) process, that ensures the LMS filter coefficients converge in expected value to α. [15%]

2 (a) Let $\{u(n)\}_{n\geq 0}$ be the input to a finite impulse response (FIR) filter of order M and $\{y(n)\}_{n\geq 0}$ be the corresponding output. The impulse response of the filter is unknown.

(i) Given the input and output sequences, minimise the following exponentially weighted least squares cost function

$$\sum_{k=0}^{n} \lambda^{n-k} \left(y(k) - \mathbf{h}^T \mathbf{u}(k) \right)^2$$

with respect to the vector $\mathbf{h}$, where $\mathbf{u}(n) = [u(n), u(n-1), \ldots, u(n-M+1)]^T$ and $0 < \lambda \leq 1$ is the *forgetting* factor. [30%]

(ii) Show that the minimiser, $\mathbf{h}(n)$, can be expressed as

$$\mathbf{R}(n)\mathbf{h}(n) = \mathbf{p}(n)$$

where the matrix $\mathbf{R}(n)$ and vector $\mathbf{p}(n)$ are to be defined. [5%]

(iii) Express $\mathbf{R}(n)$ and $\mathbf{p}(n)$ in terms of their values at time $n-1$. [5%]

(iv) For $M = 1$, show that the scalar valued minimiser can be expressed recursively as

$$h(n) = h(n-1) + K(n)u(n)\left(y(n) - u(n)h(n-1)\right)$$

where the gain $K(n)$ is to be defined. [20%]

(b) Consider the following AR(1) process

$$x(n) = \beta x(n-1) + w(n)$$

where $|\beta| < 1$ and $w(n)$ are independent and identically distributed zero-mean random variables with variance σ^2.

(i) Given $x(-1), x(0), \ldots, x(n)$, solve for the exponentially weighted least squares estimate of β. [20%]

(ii) For $\lambda = 1$, obtain the limit of this estimate as n tends to infinity. [10%]

(iii) Show that this limit is β. [10%]

3 Consider the following model for a complex frequency component observed in noise

$$x_n = a \exp(i\omega_0 n) + e_n$$

where a is a *real* scalar amplitude, ω_0 a real frequency and $\{e_n\}$ is an independent *complex* Gaussian noise sequence. This means that the real and imaginary parts of e_n are independent zero mean Gaussian variables with identical variance σ^2. Thus the probability density of e_n is

$$\frac{1}{2\pi\sigma^2} \exp\left(\frac{-|e_n|^2}{2\sigma^2}\right).$$

(a) Show that the conditional probability density of the data sequence $x_0, \ldots, x_{N-1}$ given a, i.e. $p(x_0, \ldots, x_{N-1}|a)$, is

$$\prod_{k=0}^{N-1} \frac{1}{2\pi\sigma^2} \exp\left(\frac{-|x_k - a\exp(i\omega_0 k)|^2}{2\sigma^2}\right).$$

[5%]

(b) Assuming ω_0 is known, show that the Maximum Likelihood solution for a is the minimiser of the quadratic cost function

$$\sum_{k=0}^{N-1} |x_k - a\exp(i\omega_0 k)|^2 .$$

[5%]

(c) Hence show that the Maximum Likelihood solution for a is the real part of

$$\frac{1}{N} \sum_{k=0}^{N-1} x_k \exp(-i\omega_0 k).$$

[40%]

[Hint: $\frac{d}{da} |x_k - a\exp(i\omega_0 k)|^2$ can be obtained by writing $|\cdot|^2$ as the sum of the squares of its real and imaginary parts.]

(d) Find an expression for $E\{e_{n+k} e_n^*\}$. [10%]

(e) Find the autocorrelation function $R_{XX}[k] = E\{x_{n+k} x_n^*\}$ and power spectrum of x_n. [30%]

(f) Explain carefully why the periodogram estimate of the power spectrum could be used to estimate ω_0 if it were unknown. [10%]

4 (a) Describe the *parametric* approach to power spectrum estimation, including a brief description of how the spectrum is estimated. [25%]

(b) You are given the scalar data points $x_0, \ldots, x_{N-1}$. We may fit an AR(P) model to the data by minimising the following mean square prediction error for this data sequence

$$\sum_{n=P}^{N-1} \left(x_n - \mathbf{a}^T \mathbf{x}_{n-1} \right)^2$$

where $\mathbf{a} = [a_1, \ldots, a_P]^T$ are the AR model coefficients and $\mathbf{x}_k = [x_k, \ldots, x_{k-P+1}]^T$.

(i) Explain why the sum begins at P. [5%]

(ii) Show that the $\mathbf{a}$ that minimises the mean square prediction error satisfies

$$\sum_{n=P}^{N-1} x_n \mathbf{x}_{n-1} = \left(\sum_{n=P}^{N-1} \mathbf{x}_{n-1} \mathbf{x}_{n-1}^T \right) \mathbf{a}.$$

[15%]

(iii) Explain how these simultaneous equations for $a_1, \ldots, a_P$ relate to the Yule-Walker equations for fitting the AR(P) model. [5%]

(iv) How do you estimate the variance of the driving noise of the AR(P) model? [5%]

(c) Let

$$x_n = w_n + w_{n-2}$$

where w_n is a sequence of independent random variables with zero mean and unit variance.

(i) Obtain the power spectrum of $\{x_n\}$. [10%]

(ii) Compute the expected value of the periodogram estimate from samples $x_0, x_1, \ldots, x_{N-1}$ of the process. [35%]

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