EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 27 April 2021   1.30 to 3.10

Module 4F7

STATISTICAL SIGNAL ANALYSIS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) Let $\mathbf{X} = [X_1, \ldots, X_p]^{\mathrm{T}}$ be a $p \times 1$ random vector with mean $\mathbf{b} = [b_1, \ldots, b_p]^{\mathrm{T}}$. Let $\mathbf{Y} = [Y_1, \ldots, Y_q]^{\mathrm{T}}$ be a $q \times 1$ random vector with mean $\mathbf{m}$ and

$$\mathrm{Cov}\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{pmatrix} \mathbb{E}(\mathbf{X}\mathbf{X}^{\mathrm{T}}) - \mathbf{b}\mathbf{b}^{\mathrm{T}} & \mathbb{E}(\mathbf{X}\mathbf{Y}^{\mathrm{T}}) - \mathbf{b}\mathbf{m}^{\mathrm{T}} \\ \mathbb{E}(\mathbf{Y}\mathbf{X}^{\mathrm{T}}) - \mathbf{m}\mathbf{b}^{\mathrm{T}} & \mathbb{E}(\mathbf{Y}\mathbf{Y}^{\mathrm{T}}) - \mathbf{m}\mathbf{m}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}.$$

(i) A linear estimate of X_1 using $\mathbf{Y}$ is

$$\widehat{X}_1 = b_1 + [h_1, \ldots, h_q]\,(\mathbf{Y} - \mathbf{m})\,.$$

Find the constants $h_1, \ldots, h_q$ that minimise the mean square error $\mathbb{E}\left\{(X_1 - \widehat{X}_1)^2\right\}$. [20%]

(ii) Find the minimum mean square error. [10%]

(iii) Find the matrix H such that the estimate

$$\widehat{\mathbf{X}} = \mathbf{b} + H\,(\mathbf{Y} - \mathbf{m})$$

minimises $\sum_{i=1}^{p} \mathbb{E}\left\{(X_i - \widehat{X}_i)^2\right\}$. [30%]

(b) Let X_k be the random process

$$X_k = aX_{k-1} + W_k, \qquad k = 1, 2, \ldots$$

where $X_0 \sim \mathcal{N}(0, \sigma_0^2)$ and $W_1, W_2, \ldots$ are independent $\mathcal{N}(0, \sigma_w^2)$ random variables. A noisy observation of $X_1, X_2, \ldots$ is

$$Y_k = X_k + V_k$$

where $V_1, V_2, \ldots$ are independent $\mathcal{N}(0, \sigma_v^2)$ random variables.

(i) Assume $\mathbb{E}(X_k^2) = \sigma_0^2$ for all k. Find a. [10%]

(ii) Find Σ_{xx}, Σ_{xy} and Σ_{yy}. [30%]

2 Let $X_0, X_1, \ldots$ be a Markov chain with values in $\{1, 2, \ldots\}$ with the following transition probabilities

$$p_{i,j} = \Pr(X_{n+1} = j | X_n = i) = \begin{cases} a & \text{if } j = i + 1, \\ b & \text{if } j = i - 1, \\ 1 - (a + b) & \text{if } j = i, \end{cases}$$

when $X_n = i > 1$. Let $p_{1,2} = a$ and $p_{1,1} = 1 - a$.

(a) Assume $a < b$. Using the equation $v_j = \sum_i v_i p_{i,j}$, express v_k (for any $k > 1$) as a function of v_1 and find v_1. [15%]

(b) Let $Y_n \geq 0$ be the observed process with conditional probability density function

$$p(y_n | x_n) = \lambda x_n \exp(-\lambda x_n y_n), \qquad y_n \geq 0$$

where $\lambda > 0$ is a fixed constant.

(i) Let $\pi_n(i) = \Pr(X_n = i | y_1, \ldots, y_{n-1})$. Find $\bar{\pi}_n(i) = \Pr(X_n = i | y_1, \ldots, y_{n-1}, y_n)$ and $\pi_{n+1}(j)$ for all j. [20%]

(ii) Given a sequence of T observations, let $\beta_{n+1}(j) = p(y_{n+1}, \ldots, y_T | X_{n+1} = j)$. Find $\beta_n(i)$. [20%]

(iii) Find $\Pr(X_n = i | y_1, \ldots, y_T)$. [10%]

(c) Given a sequence of values $x_1, y_1, \ldots, x_T, y_T$, find the value of λ that maximises $p(x_1, y_1, \ldots, x_T, y_T | x_0)$. [20%]

(d) Assume the constants a and b in the definition of $p_{i,j}$ are known. Describe carefully the Expectation-Maximisation algorithm for finding the value of λ that maximises $p(y_1, \ldots, y_T | x_0)$. [15%]

3 Let X_n be a real valued random process

$$X_n = X_{n-1} + W_n, \qquad n = 1, 2, \ldots$$

where $W_1, W_2, \ldots$ are independent zero mean random variables with common variance σ_w^2. Assume X_0 is random with mean μ_0 and is independent of $\{W_1, W_2, \ldots\}$.

A noisy observation of $X_1, X_2, \ldots$ is

$$Y_n = X_n + V_n, \qquad n = 1, 2, \ldots$$

where $V_1, V_2, \ldots$ are independent zero mean random variables with common variance σ_v^2.

(a) Let $\widehat{X}_{n-1}$ be an *unbiased* estimate of X_{n-1} using $Y_1, \ldots, Y_{n-1}$. Upon receiving Y_n, an estimate of X_n is

$$\widehat{X}_n = L_n \widehat{X}_{n-1} + K_n Y_n$$

where K_n and L_n are constants. Find an expression for L_n in terms of K_n so that $\widehat{X}_n$ is unbiased. [10%]

(b) Let $\sigma_{n-1}^2 = \mathbb{E}\left\{(\widehat{X}_{n-1} - X_{n-1})^2\right\}$ and consider the estimate of X_n

$$\widehat{X}_n = \widehat{X}_{n-1} + G_n \left(Y_n - \widehat{X}_{n-1}\right).$$

(i) Let $e_n = X_n - \widehat{X}_n$ and find an equation relating e_n to e_{n-1}. [10%]

(ii) Find the value of the gain G_n that minimises the mean square error $E\left\{(\widehat{X}_n - X_n)^2\right\}$. Carefully detail your derivation and justify your answer for $\sigma_v = 0$. [25%]

(c) Assume $\sigma_w^2 = 0$, which implies $X_n = X_0$ for all n.

(i) Consider the estimate of X_n

$$\widehat{X}_n = \widehat{X}_{n-1} + \frac{\sigma_{n-1}^2}{\sigma_{n-1}^2 + \sigma_v^2}\left(Y_n - \widehat{X}_{n-1}\right) = \widehat{X}_{n-1} + H_n\left(Y_n - \widehat{X}_{n-1}\right).$$

Find σ_n^2/σ_v^2. [20%]

(ii) Find the mean square error of the sample mean estimate

$$\widehat{X}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

[10%]

(iii) Assuming $\sigma_0^2 = \sigma_v^2$, conclude which estimate of X_0, the one in part (c)(i) or part (c)(ii), is better. [20%]

(iv) State the relationship between the gains G_n and H_n. [5%]

4 Consider the following random process driven by Gaussian noise

$$X_{k+1} = \alpha X_k + W_{k+1}, \quad k = 0, 1, \ldots$$

where α is a non-random constant and $W_1, W_2, \ldots$ are independent $\mathcal{N}(0,1)$ random variables, i.e. have zero mean and unit variance. Assume $X_0 = 0$.

(a) Let $x_{1:k}$ denote the sequence $x_1, \ldots, x_k$. Find α that maximises the joint *probability density function* (pdf) $p(x_{1:k})$. [20%]

(b) The random process X_k is observed indirectly through the random process

$$Y_k = |X_k| + V_k, \quad k = 1, 2, \ldots$$

where $V_1, V_2, \ldots$ are independent and identically distributed random variables with common probability density function $f(v)$.

(i) Find the conditional pdf $p(y_k|x_k)$ in terms of $f(v)$. [10%]

(ii) Given the conditional pdf $p(x_k|y_{1:k})$, find $p(x_{k+1}|y_{1:k+1})$. [10%]

(c) Let $X^1_{1:k}, X^2_{1:k}, \ldots, X^N_{1:k}$ be N samples of $(x_1, \ldots, x_k)$ and $w^1_k, w^2_k, \ldots, w^N_k$ be N random variables, known as *weights*, such that

$$\mathbb{E}\left\{h(X^i_{1:k})w^i_k\right\} = \int \ldots \int \int h(x_{1:k}) p(x_{1:k}, y_{1:k}) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_k \qquad (1)$$

where $h(x_{1:k})$ is any real-valued function. Find the function $h(x_{1:k})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \log p(y_{1:k}) = \int h(x_{1:k}) p(x_{1:k}|y_{1:k}) \mathrm{d}x_{1:k}$$

and find the importance sampling estimate of the value of α such that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \log p(y_{1:k}) = 0.$$

[30%]

(d) Let $J_1, \dots, J_N$ be independent and identically distributed random variables with probability mass function

$$\Pr(J = j) = \frac{w_k^j}{\sum_{i=1}^N w_k^i}.$$

(i) Show that the *resampled* estimate

$$\frac{1}{N}\frac{w_k}{N}\left(h(X_{1:k}^{J_1}) + \dots + h(X_{1:k}^{J_N})\right)$$

where $w_k = \sum_{i=1}^N w_k^i$ has the same expected value in equation (1) of part (c). [20%]

(ii) Using sequential importance sampling with resampling and the weighted samples of part (c), find an estimate of

$$\int g(x_{1:k+1}) p(x_{1:k+1}|y_{1:k+1}) \mathrm{d}x_{1:k+1}.$$

[10%]

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