Version JL/3

EGT3
ENGINEERING TRIPOS PART IIB

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Tuesday 30 April 2024 2 to 3.40

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**Module 4F8**

**IMAGE PROCESSING AND IMAGE CODING**

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number* ***not*** *your name on the cover sheet.*

**STATIONERY REQUIREMENTS**
Single-sided paper.

**SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM**
CUED approved calculator allowed.
Engineering Data Book

**10 minutes reading time is allowed for this paper at the start of the exam.**

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.**

**You may not remove any stationery from the Examination Room.**

1 (a) If the 2D Fourier transform of a function $f(u_1, u_2)$ is $F(\omega_1, \omega_2)$, find the Fourier transform of the translated function $f(u_1 - a, u_2 - b)$, where $a$ and $b$ are constants. [15%]

(b) Similarly, if the $(u_1, u_2)$ coordinate system is rotated by a constant angle $\phi$ to give new coordinates

$$u_1' = u_1 \cos\phi + u_2 \sin\phi, \quad u_2' = -u_1 \sin\phi + u_2 \cos\phi$$

find the Fourier transform of the rotated function $f(u_1', u_2')$. In particular show that this new transform corresponds to a rotated version of the original transform in the $(\omega_1, \omega_2)$ plane. Is the sense of rotation of the transform in the $(\omega_1, \omega_2)$ plane the same as, or the negative of, the sense of rotation of the image in the $(u_1, u_2)$ plane? [20%]

(c) Now consider an image consisting of a thin line along the $u_1$-axis in the $(u_1, u_2)$ plane, which we can approximate as being the delta function $f(u_1, u_2) = \delta(u_2)$. Note we are taking the image plane to be infinite, and we are assuming the line is infinitely long.

Find the Fourier transform of this line in the $(\omega_1, \omega_2)$ plane, and show that it is itself another 'line' in the delta function approximation. What is the angle between this line in the $(\omega_1, \omega_2)$ plane and the original line in the $(u_1, u_2)$ plane? [20%]

(d) Take the general line $u_2 = mu_1 + c$ in the $(u_1, u_2)$ plane. Note that we can represent this by the function

$$f(u_1, u_2) = \delta\left(u_2 - (mu_1 + c)\right)$$

(i) By carrying out a direct Fourier transform, show that this transforms to a function with values confined to another line in the $(\omega_1, \omega_2)$ plane. Find the explicit equation for the location of this new line and an expression for how the phase varies along the line. Both answers should be given in terms of $\omega_1$, $\omega_2$, $m$ and $c$. [10%]

(ii) Show that the answers to Part (d)(i) can also be derived by rotating and translating the result for the transform of a line along the $u_1$-axis that you found in Part (c). [10%]

(e) Explain, qualitatively, the effects on the transform if the line being transformed has only a finite, rather than infinite length. You may still assume the image plane itself is infinite. [5%]

(f) Consider the image in Fig. 1, in which the background (white) has image values of 0, and the lines themselves (black) have image values of 1.

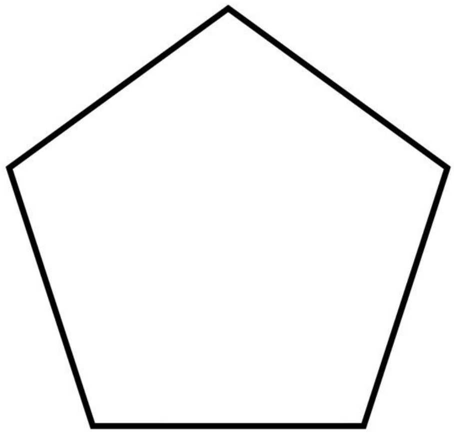

Fig. 1

Draw a sketch of what you think the amplitude of the Fourier transform of this image would look like, giving written justifications for the main features. [20%]

2 (a) Consider the ideal bandpass filter shown in Fig. 2, with $H = 1$ in the shaded regions and $H = 0$ otherwise. Sampling is carried out on a rectangular grid with spacings of $\Delta_1$ and $\Delta_2$ in the $u_1$ and $u_2$ directions respectively.

Using standard results or otherwise, form the ideal impulse response, $h(n_1\Delta_1, n_2\Delta_2)$, of this filter by taking an *alternative ideal bandpass filter* and letting $\Omega_{L1}$ be zero (where $\Omega_{L1}$ is the lower limit in the $\omega_1$ direction).

Check that this gives the same result as forming the filter by subtracting two lowpass filters and describe these lowpass filters. [25%]

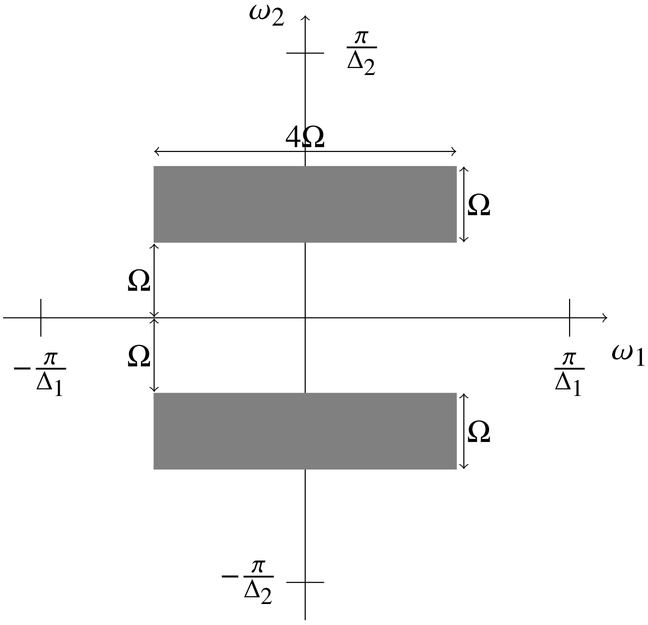


Fig. 2

(b) Assume that an observed image $\mathbf{y}$ can be modelled as a linear distortion, $\mathbf{L}$, of the true image $\mathbf{x}$ plus additive noise $\mathbf{d}$.

(i) If $\mathbf{L}$ is known and we neglect noise, explain how the *inverse filter* can be used to estimate the true image. [15%]

(ii) More sophisticated deconvolution techniques will attempt to estimate the *posterior*, $P(\mathbf{x}|\mathbf{y})$, via Bayes theorem, $P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{y}|\mathbf{x})P(\mathbf{x})$.

Give the form of the $S$ in the *prior*, $P(\mathbf{x}) = e^{\alpha S}$, for the *Maximum Entropy* deconvolution method and derive, by differentiation, the global maximum of $S$. Explain why this is useful. Describe all terms you use in your expressions and any assumptions made. [20%]

 (cont.

(c) (i) $G(\omega_1, \omega_2)$ is the Fourier transform of an image $g$, and $G_s(\omega_1, \omega_2)$, is the Fourier transform of the corresponding *sampled* image $g_s$. Using the form of $G_s$ in terms of $G$, explain the phenomenon of *aliasing*. [15%]

(ii) A $512 \times 256$ unaliased image $x(u_1, u_2)$ has a Discrete Fourier Transform $X(\omega_1, \omega_2)$ which is a $512 \times 256$ array of complex numbers. We find that the highest frequencies of $X$ occur at the points $\pm(10, 120)$ and $\pm(200, 20)$, where we are taking $(0, 0)$ to be in the centre of the $512 \times 256$ array.

Sketch these frequencies in the $(\omega_1, \omega_2)$ plane. If we now wish to downsample our image to an $n_1 \times n_2$ image, find the minimum values of $n_1$ and $n_2$ that will still produce an *unaliased* image. [25%]

3 (a) Sketch the main blocks of an image coding/decoding system. [5%]

(b) The Haar matrix $T$, given below, is applied (by acting on $2 \times 2$ blocks) to a $4 \times 4$ image $X$ to give a resulting image $Y$:

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

Evaluate $Y$ and then regroup components to form an image $Y'$, where blocks in $Y'$ consist of corresponding elements from the blocks in $Y$. Comment on the form of $Y'$. Is this what you would expect from looking at the original image $X$. [35%]

(c) Comment on the energy content of each $2 \times 2$ sub-block of $Y'$ found in Part (b). Explain, qualitatively, how the nature of the sub-blocks (or sub-bands) allow us to compress the image. [15%]

(d) If the $2 \times 2$ Haar transform is applied to a typical natural image, sketch the energies and entropies of each of the 4 sub-bands (Lo-Lo, Hi-Lo, Lo-Hi, Hi-Hi), assuming some fixed quantisation step. [15%]

(e) The *Approximate Entropy* $H_a$ of a sub-band (other than Lo-Lo) in Part (d), which has been uniformly quantised, is given by

$$H_a = \log_2 \left( \frac{2ex_0}{Q} \right)$$

Explain what the quantities $x_0$ and $Q$ are by reference to the probability density function used to approximate the entropy of the quantised samples. [15%]

(f) The $n \times n$ *Discrete Cosine Transform* (DCT) has better image compression properties than the Haar transform (for $n > 2$). Explain why this is and give a brief overview of how the properties of the DCT enable efficient computation. [15%]

4 (a) For a lowpass filter $H_0(z)$ and a highpass filter $H_1(z)$, sketch a 1D 4-level binary filter tree. If our input vector has $N$ samples, and $f_s$ is the sampling frequency, indicate the sizes and bandwidths of each of the highpass outputs at each level. [20%]

(b) Sketch the *inverse tree* which takes the output of the binary tree in Part (a) and reconstructs the input signal using another pair of lowpass, $G_0(z)$, and highpass, $G_1(z)$, filters. For *perfect reconstruction* (PR) give the two conditions that must be satisfied by $H_0, H_1, G_0, G_1$. [15%]

(c) A lowpass product filter $P_0(z)$ is defined by:

$$P_0(z) = H_0(z)G_0(z)$$

Consider the following expressions for forming highpass filters:

$$H_1(z) = z^{-k}G_0(-z) \quad \text{and} \quad G_1(z) = z^k H_0(-z)$$

If the PR conditions in Part (b) are to be satisfied, derive the condition, $P_0(z)+P_0(-z) = 2$, that $P_0(z)$ must satisfy. What restrictions does this put on the powers of $z$ in $P_0(z)$? [15%]

(d) Define $Z = \frac{1}{2}(z+z^{-1})$. It can be shown that smoother wavelet filters require multiple factors of $(1 + Z)$. If $P_0(z)$ is given by:

$$P_0(z) = (1 + Z)^2(1 + aZ)$$

find the value of $a$ if the conditions in Part (c) are to be satisfied. Suggest a possible split of this $P_0$ to give $H_0$ and $G_0$. [20%]

(e) For the LeGall(3,5) filters given below:

$$H_0(z) = \frac{1}{2}(z + 2 + z^{-1}), \qquad H_1(z) = \frac{1}{8}z^{-1}(-z^2 - 2z + 6 - 2z^{-1} - z^{-2})$$

sketch the impulse and frequency responses for the first three levels of the filter tree. [20%]

(f) To apply the techniques described in previous parts of this question to images, we first apply the 1D filter to rows and then to columns. Sketch 2 levels of a 2D filter tree with $(H_0, H_1)$ as lowpass and highpass filters. [10%]

**END OF PAPER**

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