EGT3
ENGINEERING TRIPOS PART IIB

Tuesday 6 May 2025   2 to 3.40

Module 4F8

IMAGE PROCESSING AND IMAGE CODING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The Fourier transform of an image $g(u_1, u_2)$ (assume all values of g are real and non-negative) produces a 2D set of complex values $G(\omega_1, \omega_2)$.

(a) G will have both amplitude and phase. Discuss the importance of phase in images and explain how one might illustrate this for a given image. [10%]

(b) For an image $g(u_1, u_2)$ in the (u_1, u_2) plane, a radial scaling by a factor α is carried out via defining a new image $g'(u_1, u_2)$ by

$$g'(u_1, u_2) = g(u_1', u_2') \quad \text{with} \quad u_1' = \alpha u_1, \quad u_2' = \alpha u_2$$

where α is a positive constant.

Find an expression for the new Fourier transform $G'(\omega_1, \omega_2)$ in terms of the original, $G(\omega_1, \omega_2)$. [20%]

(c) By adopting polar coordinates (r, ϕ) in the (u_1, u_2) plane and (ω, θ) in the (ω_1, ω_2) plane, show that it is possible to obtain a 'polar' version of the 2D Fourier transform, in the form

$$G(\omega, \theta) = \int_0^{2\pi} \int_0^{\infty} g(r, \phi) f(r, \phi, \omega, \theta)\, dr\, d\phi$$

where you should determine the function $f(r, \phi, \omega, \theta)$ explicitly. [20%]

(d) An infinite image consists of a white circle on a black background. The circle is centred at the origin and has radius r_0, and the image values are $a_{\max}$ on the circle and 0 on the black background. The width of the line composing the circle is ϵ and is very much smaller than its radius (r_0), meaning that $g(r, \phi)$ can be treated as a δ-function in the r direction, centred at $r = r_0$ and with a strength $\epsilon a_{\max}$.

Find the polar Fourier transform, G, of this image. [20%]

[Note: You may assume $\int_0^{2\pi} \exp(-ja\cos\phi)d\phi = 2\pi J_0(a)$, where J_0 is a Bessel function of the first kind and a is a real number.]

(e) Using the plot of $J_0(x)$ in Fig. 1, sketch the modulus of G in the Fourier plane for $r_0 = 1$. [10%]

(f) Describe the phase of G. If the circle is translated so that its origin is at $(u_1, u_2) = (c, 0)$ to give an new image $\tilde{g}$, describe the phase of the Fourier transform of this translated image. [10%]

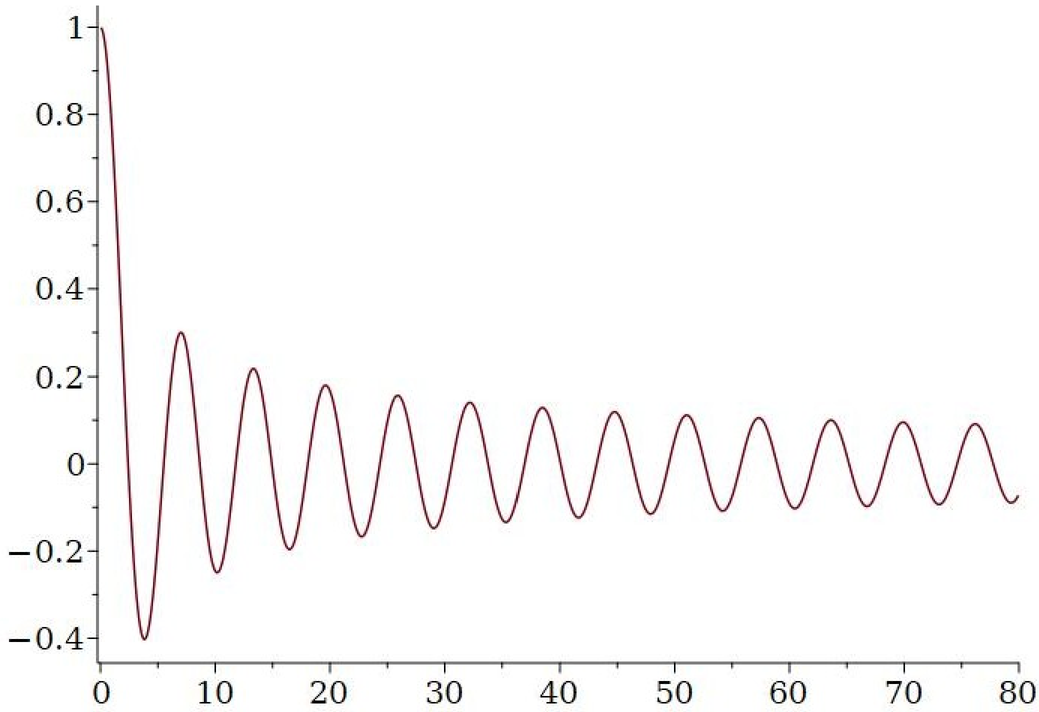

Fig. 1

(g) If the image now consists of two circles of radius 1 centred at $(c, 0)$ and $(-c, 0)$, describe the phase and amplitude of the resulting Fourier transform (there is no need to sketch). [10%]

2 (a) An image $g(u_1, u_2)$ is sampled on a rectangular grid to give a sampled image $g_s(u_1, u_2)$.

(i) Give the expression (there is no need to derive it) for the Fourier transform, $G_s(\omega_1, \omega_2)$ of this sampled image. Assume the sample spacings are Δ_1 and Δ_2 in the u_1 and u_2 directions. [15%]

(ii) Now consider an image which has a radially symmetric Fourier transform with a radial profile proportional to that given in Fig. 1 of question 1. If this image is sampled, estimate the maximum sample spacings, Δ_1 and Δ_2, possible if the aliased component is to contribute at most an amplitude of around 0.1 of its value at $\omega_1 = \omega_2 = 0$. State any assumptions made. [20%]

(b) Filtering of 2D images is the convolution of the image with an *impulse response* (IR) (or *point spread function* (PSF)).

(i) In order to design a filter with a desired zero-phase 2D frequency response, explain why *windowing* is necessary and describe the effect of windowing on the desired frequency response. [10%]

(ii) Describe two methods of forming 2D window functions from 1D window functions. [10%]

(c) An observed image $\mathbf{y}$ can be modelled as a linear distortion, $\mathbf{L}$, of the true image $\mathbf{x}$ plus additive noise $\mathbf{d}$.

(i) One method of deconvolution attempts to estimate the *posterior*, $P(\mathbf{x}|\mathbf{y})$ (the probability of the true underlying image given the observations), via Bayes' theorem, $P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{y}|\mathbf{x})P(\mathbf{x})$. Give the form of $P(\mathbf{x}|\mathbf{y})$ for the *Wiener Filter* stating any assumptions made and explaining all quantities used (you need not derive the filter). [10%]

(ii) Now assume that the Gaussian prior on the true image can be taken as

$$P(\mathbf{x}) \propto \exp\left(-\frac{1}{\lambda}\sum_i x_i^2\right)$$

where λ is a positive constant and the sum is over all pixels in the image.
In the case where there is no convolution and $\mathbf{y}$ has a noise covariance matrix which is diagonal with constant entries σ^2, write down an expression for $P(\mathbf{x}|\mathbf{y})$ and differentiate to find the optimal solution, $\hat{\mathbf{x}}$. [25%]

(iii) Confirm that your solution to part (c)(ii) is indeed the Wiener solution in part (c)(i). [10%]

3 (a) Sketch a basic *image coding system*. Label all blocks of the *encoder* and the *decoder*. Explain the purpose of each block and indicate which processes can be lossless and which cannot. [15%]

(b) Explain how to apply the 2×2 Haar matrix, $\mathbf{T}$, to a $2n \times 2n$ image X to produce an image Y. Then describe the process of rearranging Y into a number of $n \times n$ subimages. What are the properties of these subimages? [15%]

(c) Consider the 4×4 image X given below:

$$X = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

(i) Apply the 2×2 Haar transform to X, to give a 4×4 array Y. Rearrange Y into a 4×4 array Y' as described in part (b). Label the subimages of Y'. [20%]

(ii) Now apply a second Haar transform to the Lo-Lo subimage of Y'. Give the resulting image Y'', which is the final result of this 2-level Haar process. [10%]

(iii) Verify that this 2-level Haar process has preserved total energy. Has the process succeeded in compressing energy into a small number of subimages? [10%]

(d) The $n \times n$ *Discrete Cosine Transform* (DCT) has better compression properties than the Haar transform ($n > 2$). Basic JPEG uses an 8×8 DCT.

(i) Sketch the rows of the 8×8 DCT matrix, pointing out the symmetries present. [15%]

(ii) An 8×8 DCT is applied to a 64×64 image, and rearranged to form 64 8×8 subimages in the usual way. Let us label these subimages U_{ij} for $i, j = 1, \ldots, 8$. The mean entropy, H_{ij}, of the coefficients in each subimage is given approximately by

$$H_{ij} = \frac{6}{i + j - 1} \quad \text{bits/coefficient}$$

Estimate the total number of bits required to encode this image. [15%]

4 (a) Figure 2(a) shows a digital filter, with transfer function $H(z)$, preceded by a 2:1 downsampler and followed by a 2:1 upsampler. Write down an expression relating the output samples, $\hat{y}(n)$, to the input samples, $x(n)$, and the filter impulse response coefficients, $h(k)$, $k = 1 \ldots K$. Hence show that the systems in Fig. 2(b) and Fig. 2(c) are both equivalent to the system in Fig. 2(a). [30%]

[Note: You may assume that if $\hat{y}(n) = y(n)$ for n even, and $\hat{y}(n) = 0$ for n odd, then, in the z-domain, $\hat{Y}(z) = \frac{1}{2}[Y(z) + Y(-z)]$.]

(b) Given lowpass analysis and reconstruction wavelet filters, $H_0(z)$ and $G_0(z)$, show that if the highpass analysis and reconstruction filters are given by

$$H_1(z) = z^{-1}G_0(-z) \quad \text{and} \quad G_1(z) = zH_0(-z)$$

the anti-aliasing condition, $G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$ (one of the *perfect reconstruction* conditions), is satisfied. [10%]

(c) A lowpass product filter is defined as $P(z) = H_0(z)G_0(z)$, and the highpass filters, G_1 and H_1, are formed as in part (b). Show that for this product filter to satisfy the other *perfect reconstruction* condition ($G_0(z)H_0(z) + G_1(z)H_1(z) = 2$), $P(z)$ must satisfy

$$P(z) + P(-z) = 2$$

What does this mean for even and odd powers of z? If, in addition, the condition of *zero phase* is imposed, what additional constraint does this impose on the form of $P(z)$? [20%]

(d) The lowpass analysis and reconstruction filters for the LeGall (3,5) wavelets are given by:

$$\text{Analysis filter:} \quad H_0(z) = \frac{1}{2}(z + 2 + z^{-1})$$

$$\text{Reconstruction filter:} \quad G_0(z) = \frac{1}{8}(-z^2 + 2z + 6 + 2z^{-1} - z^{-2})$$

If the highpass analysis and reconstruction LeGall filters are formed as given in part (b), verify that the LeGall wavelets satisfy both perfect reconstruction conditions. [20%]

(e) The analysis filters in part (d) are used in a one-dimensional two-level wavelet transform system. Using results from part (a) or otherwise, derive expressions for the z-transfer functions from the input of the transform to the two level-2 outputs, $H_{00}(z)$ and $H_{01}(z)$. [20%]

(cont.

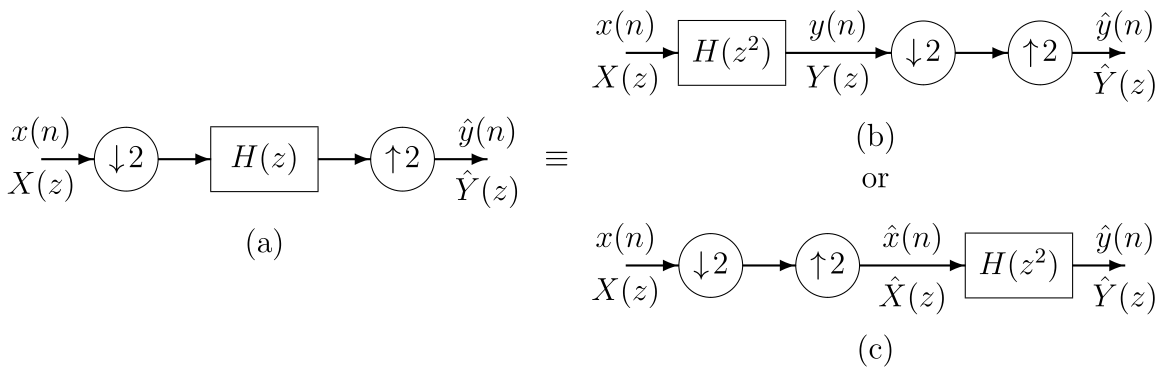

Fig. 2

END OF PAPER

THIS PAGE IS BLANK