4M12 2022 Crib

① (a) $$(\underset{\sim}{c}_g)_i = \frac{\partial \varpi}{\partial k_i}$$ where $\varpi = \varpi(\underset{\sim}{k})$ is the dispersion relationship.

- $\underset{\sim}{c}_g(\underset{\sim}{k}_0)$ is the velocity at which wavepackets of dominant wavevector $\underline{k}_0$ move
- $\underset{\sim}{c}_g(\underset{\sim}{k})$ is the velocity at which one must travel to keep seeing waves of wavevector $\underset{\sim}{k}$
- $\underset{\sim}{c}_g(\underset{\sim}{k})$ is the velocity at which energy held in waves of wavevector $\underset{\sim}{k}$ propagates.

(b) (i) Look for solutions of form ~~[illegible]~~

$$u_z \sim \exp[j(\underset{\sim}{k}\cdot\underset{\sim}{x} - \varpi t)]$$

$$+\varpi^2 k^2 - (2\Omega)^2 k_z^2 - N^2(k_x^2 + k_y^2) = 0$$

$$\Rightarrow \quad \varpi^2 = (2\Omega)^2 \frac{k_z^2}{k^2} + N^2 \frac{k_x^2 + k_y^2}{k^2}$$

$$\Rightarrow \quad \varpi^2 = N^2 + [(2\Omega)^2 - N^2]\frac{k_z^2}{k^2}, \quad f = (2\Omega)^2 - N^2$$

(ii)
$$\begin{cases} 2\varpi (c_g)_x = -[(2\Omega)^2 - N^2]\, k_z^2 \dfrac{2k_x}{k^4} \\ 2\varpi (c_g)_y = -[(2\Omega)^2 - N^2]\, k_z^2 \dfrac{2k_y}{k^4} \\ 2\varpi (c_g)_z = -[(2\Omega)^2 - N^2] \dfrac{2k_z^3 - 2k_z k^2}{k^4} \end{cases}$$

$$\Rightarrow \quad \cancel{2}\varpi \underset{\sim}{c}_g = -[(2\Omega)^2 - N^2]\frac{\cancel{2}}{k^4}[k_x k_z^2,\ k_y k_z^2,\ -k_z(k_x^2 + k_y^2)]$$

$$\Rightarrow \quad \underset{\sim}{c}_g = \frac{((2\Omega)^2 - N^2)}{\varpi k^4}[-k_z^2 k_x,\ -k_z^2 k_y,\ (k_x^2 + k_y^2)k_z]$$

$$\Rightarrow \quad \underset{\sim}{c}_g = [(2\Omega)^2 - N^2]\frac{k_\perp^2 k_{\#} - k_z^2 k_\perp}{\varpi k^4}$$

$$(g = (2\Omega)^2 - N^2)$$

(iii) If $2\Omega = N$, $\frac{\partial^2}{\partial t^2}\nabla^2 u_z + (2\Omega)^2 \nabla^2 u_z = 0$

$\Rightarrow \frac{\partial^2}{\partial t^2} u_z + (2\Omega) u_z = 0$ No wave motion, only oscillation.

① Cont.

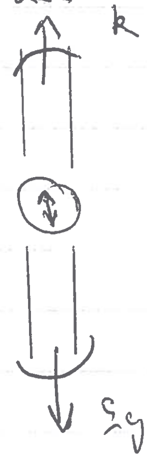
(c)(i) $N = 0 \Rightarrow \begin{cases} \varpi = \pm 2\Omega \dfrac{k_z}{k} \\ \\ \underset{\sim}{c}_g = (2\Omega)^2 \dfrac{k_\perp^2 \underset{\sim}{k}_{\parallel} - k_z^2 \underset{\sim}{k}_\perp}{\varpi k^4} \end{cases}$

$$\Rightarrow \underset{\sim}{c}_g = \pm 2\Omega \frac{k_\perp^2 \underset{\sim}{k}_{\parallel} - k_z^2 \underset{\sim}{k}_\perp}{k_z k^3} = \pm 2\Omega \frac{k_\perp^2 \hat{\underset{\sim}{e}}_z - k_z \underset{\sim}{k}_\perp}{k^3}$$

$$\left[G = \frac{k_\perp^2 \hat{\underset{\sim}{e}}_z - k_z \underset{\sim}{k}_\perp}{k^3} \right]$$

(ii) $\varpi \ll \Omega \Rightarrow k_z \simeq 0 \Rightarrow \underset{\sim}{c}_g \simeq \pm 2\Omega \dfrac{\hat{\underset{\sim}{e}}_z}{k}$

dispersion is along the z axis only



$\dot{\varpi} = 3\Omega$ – no waves as $\varpi \leq 2\Omega$.

② (a) Expect a self-similar soln. if there is no imposed geometric length scale.

$l = f(\alpha, t)$, Parameters $= P = 3$

l: m, α: $m^2 s^{-1}$, t: s. Dimensions $= D = 2$

Number of dimensionless groups $= P - D = 1$

By inspection, $\Pi = \dfrac{l}{\sqrt{\alpha t}}$

There is no other group for Π to depend on, so

$\Pi =$ constant $\Rightarrow$ $l \sim \sqrt{\alpha t}$

(b) (i) $\dfrac{\partial T}{\partial t} = \alpha \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial}{\partial r} T\right) = \alpha\left[\dfrac{\partial^2 T}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial T}{\partial r}\right]$

Compare $\dfrac{\partial \theta}{\partial t} = \alpha \dfrac{\partial^2 \theta}{\partial r^2} \Rightarrow r\dfrac{\partial T}{\partial t} = \alpha \dfrac{\partial}{\partial r}\left(\dfrac{\partial}{\partial r} rT\right)$

$\Rightarrow \dfrac{\partial T}{\partial t} = \dfrac{\alpha}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r} + T\right)$

$= \alpha\left(\dfrac{\partial^2 T}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial T}{\partial r}\right)$

(same)

(ii) $\theta = A f\left(\dfrac{r}{2\sqrt{\alpha t}}\right) = A f(\eta)$

$\dfrac{\partial \theta}{\partial t} = A f'(\eta)\left[-\dfrac{1}{2}\dfrac{\eta}{t}\right] = -\dfrac{1}{2} A f'(\eta)\dfrac{\eta}{t}$

$\alpha \dfrac{\partial^2 \theta}{\partial r^2} = \alpha A f''(\eta)\dfrac{1}{4\alpha t} = \dfrac{1}{4} A f''(\eta)\dfrac{1}{t}$

$\dfrac{\partial \theta}{\partial t} = \alpha \dfrac{\partial^2 \theta}{\partial r^2} \Rightarrow \dfrac{1}{4}\dfrac{A}{t} f''(\eta) = -\dfrac{1}{2}\dfrac{A}{t}\eta f'(\eta)$

$\Rightarrow$ $f'' + 2\eta f' = 0$

(iii) Integrate once $f' = B e^{-\eta^2}$, $B =$ const.

Second integration : $f = C + B\int^{\eta} e^{-\eta^2} d\eta$

(b) (iii) cont.

$$f(\infty) = 0 \Rightarrow C + B\underbrace{\int_0^\infty e^{-\eta^2} d\eta}_{\sqrt{\pi}/2} = 0$$

$$\Rightarrow B = -\frac{2}{\sqrt{\pi}} C$$

$$\Rightarrow f = C\left[1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta\right]$$

But $f(0) = 1$ so $C = 1$

$$\Rightarrow \underline{\underline{f = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta}} \Rightarrow \underline{\underline{T = \frac{A}{r}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2} d\eta\right]}}$$

(iv) $\dot{q} = -k\frac{\partial T}{\partial r}$, $\frac{\partial T}{\partial r} = -\frac{A}{r^2} f(\eta) - \frac{2}{\sqrt{\pi}}\frac{A}{r} e^{-\eta^2}\frac{1}{2\sqrt{\alpha t}}$

$$\dot{Q} = 4\pi r^2 \dot{q} = 4\pi k A\left[f(\eta) + \frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}\right] = 4\pi k A \text{ at } \eta = 0$$

$$\Rightarrow \underline{\underline{A = \dot{Q}/4\pi k}}$$

(c) Proof by contradiction. Let T_1 and T_2 be 2 distinct solutions.

Let $\Phi = T_1 - T_2$, so $\nabla^2\Phi = 0$, $\Phi = 0$ on surface.

$$\nabla\cdot[\Phi\nabla\Phi] = \Phi\nabla^2\Phi + (\nabla\Phi)^2 = (\nabla\Phi)^2$$

$$\Rightarrow \oint_S \Phi\nabla\Phi\cdot d\underset{\sim}{S} = \int_V (\nabla\Phi)^2 dV$$

(Gauss)

But $\Phi = 0$ on $S \Rightarrow \underline{\underline{\int_V (\nabla\Phi)^2 dV = 0}}$

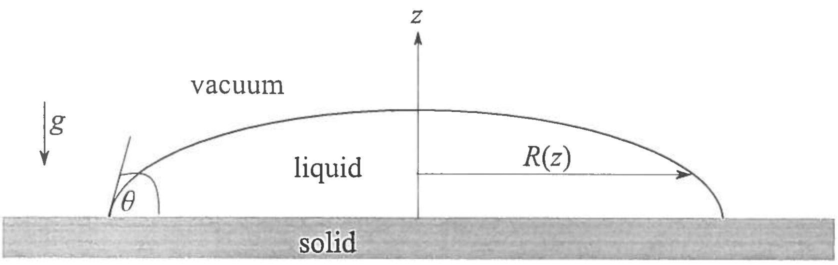
$\Rightarrow \nabla\Phi = 0$ for all $\underset{\sim}{x}$

$\Rightarrow \Phi = \text{const.}$

$\Rightarrow \Phi = 0$ (as $\Phi = 0$ on S)

$\Rightarrow \underline{\underline{T_1 = T_2}}$

3 An axisymetric drop of liquid of density ρ sits on a planar surface in a gravitational field g, as shown below.



The system contains three interfaces, each of which costs an interfacial energy γ_i per unit area, with $i = 1$ being the liquid-vacuum, $i = 2$ liquid-solid and $i = 3$ solid-vacuum. Thus, if each interface has an area A_i, the total interfacial energy is

$$E_{int} = A_1\gamma_1 + A_2\gamma_2 + A_3\gamma_3.$$

The drop's shape minimizes its gravitational and interfacial energy, while having fixed volume V.

(a) The shape is described by the function $R(z)$, with $z = 0$ being the solid, and $z = h$ being the top of the drop. $R(z)$ minimizes a functional of the form

$$E(R) = \int_0^h I(R, R', z)\mathrm{d}z + AR(0)^2 + B.$$

Find expressions for the integrand, $I(R, R', z)$ and the constant A. You do not need to find the constant B. [20%]

Interfacial energy - the solid vacuum interface is a surface or revolution, the solid-liquid is a disk, and the solid-vacuum is an annulus.

$$E_{int}(R) = \gamma_1 \int_0^h 2\pi R\sqrt{1 + R'(z)^2}\mathrm{d}z + \gamma_2(\pi R(0)^2) + \gamma_3(\pi L^2 - \pi R(0)^2)$$

Volume - just a volume of revolution.

$$V(R) = \int_0^h \pi R^2 \mathrm{d}z$$

Gravitational energy, using mgh, is

$$E_g(R) = \int_0^h \rho g z \pi R^2 \mathrm{d}z$$

Using the method of Lagrange multipliers to implement the constraint of total volume, we get the modified functional

$$E(R) = \int_0^h \gamma_1 2\pi R\sqrt{1 + R'(z)^2} + \rho g z \pi R^2 + \lambda(\pi R^2 - V)\mathrm{d}z + (\gamma_2 - \gamma_3)\pi R(0)^2 + B$$

so $I(R, R', z) = \gamma_1 2\pi R\sqrt{1 + R'(z)^2} + \rho g z\pi R^2 + \lambda(\pi R^2 - V)$ *and* $A = (\gamma_2 - \gamma_3)\pi$.

(b) The drop makes a contact angle θ with the solid, as marked on the diagram. Use the directional derivative of $E(R)$ to find $\cos(\theta)$. [30%]

Taking the directional derivative in the direction of $\delta R(z)$, *we get*

$$DE(z)[\delta R] = \int_0^h \frac{\partial I}{\partial R}\delta R + \frac{\partial I}{\partial R'}\delta R'\mathrm{d}z + (\gamma_2 - \gamma_3)\pi 2R(0)\delta R(0)$$

Integrating by parts,

$$DE(z)[\delta R] = \int_0^h \frac{\partial I}{\partial R}\delta R - \frac{d}{dz}\left(\frac{\partial I}{\partial R'}\right)\delta R\mathrm{d}z + \left[\left(\frac{\partial I}{\partial R'}\right)\delta R\right]_0^h + (\gamma_2 - \gamma_3)\pi 2R(0)\delta R(0)$$

We need this to vanish for all directions δR. *Requiring the integral term to vanish would give the standard EL equation for* $R(z)$, *but that isn't the question. Instead, we ask that the* $\delta R(0)$ *boundary term vanishes, giving*

$$(\gamma_2 - \gamma_3)\pi 2R(0) = \left.\frac{\partial I}{\partial R'}\right|_{z=0}$$

Inserting the definition of I, *gives*

$$\gamma_1 2\pi R(0)R'(0)\frac{1}{\sqrt{1 + R'(0)^2}} + (\gamma_2 - \gamma_3)2\pi R(0) = 0$$

And hence

$$R'(0)\frac{1}{\sqrt{1 + R'(0)^2}} = \frac{\gamma_2 - \gamma_3}{\gamma_1}$$

Identifying that $R'(0) = \cot(\theta)$, *this gives*

$$\cos\theta = \frac{\gamma_2 - \gamma_3}{\gamma_1}.$$

(c) Assuming gravity is negligible:

(i) Show that $R(z)$ obeys the differential equation

$$\lambda R = \frac{1}{\sqrt{1 + R'^2}}$$

 (cont.

where λ is a constant. [30%]

Neglecting gravity removes the z dependence from the integrand, so we may use the Beltrami form of the EL equations to get

$$\frac{d}{dz}\left(I - R'\frac{\partial I}{\partial R'}\right) = 0$$

Integrating, we get

$$\left(I - R'\frac{\partial I}{\partial R'}\right) = c$$

Then inserting the definition of I:

$$\left(\lambda\pi R^2 + \gamma_1 2\pi R\sqrt{1+R'^2} - R'\frac{\gamma_1 2RR'}{\sqrt{1+R'^2}}\right) = c$$

$$\left(\lambda\pi R^2 + \gamma_1 2\pi\frac{R}{\sqrt{1+R'^2}}\right) = c$$

$$\pi R\left(\lambda R + \frac{2\gamma_1}{\sqrt{1+R'^2}}\right) = c$$

However, looking at the top boundary, we have $R = 0$ meaning that $c = 0$. Redefining the constant λ then leads to the stated result.

(ii) Find the functional form of $R(z)$, and give a geometric interpretation. You do not need to determine the value of any constants in your solution. [20%]

Solving first for R', we get

$$R'^2 = \left(\frac{1}{\lambda R}\right)^2 - 1$$

Taking the negative square root, as $R'(z)$ is negative for the drop

$$R' = -\sqrt{\left(\frac{1}{\lambda R}\right)^2 - 1}$$

$$\int \frac{dR}{\sqrt{\left(\frac{1}{\lambda R}\right)^2 - 1}} = -\int \mathrm{d}z$$

$$\int dR\sqrt{\frac{R^2}{\left(\frac{1}{\lambda}\right)^2 - R^2}} = -\int \mathrm{d}z$$

We may integrate this directly to get

$$\sqrt{\left(\frac{1}{\lambda}\right)^2 - R^2} = z + d$$

where d is a constant of integration. Squaring and rearranging we get

$$R^2 + (z + d)^2 = \left(\frac{1}{\lambda}\right)^2$$

This is the equation of a circle of radius $\frac{1}{\lambda}$ *and centered at* $z = -d$*, so the droplet is forming a spherical cap.*

[Hint: You may regard h as a known quantity, so you do not need to minimizing over h.]

4 A liquid crystal is a fluid of rod shaped molecules, in which the molecules align along a direction described by the unit vector, $\hat{\mathbf{n}}$. If the alignment direction $\hat{\mathbf{n}}(\mathbf{x})$ is spatially varying then the liquid crystal has an Frank elastic energy,

$$E_F(\hat{\mathbf{n}}) = \int \frac{1}{2}K_1 (\nabla \cdot \hat{\mathbf{n}})^2 + \frac{1}{2}K_2 (\hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}})^2 + \frac{1}{2}K_3 |\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}|^2 \, dV.$$

where K_1, K_2 and K_3 are known as the Frank constants, and the integral is over the volume of the liquid crystal.

(a) Express the integrand of the Frank energy using index notation and the summation convention. [15%]

$$(\nabla \cdot \hat{\mathbf{n}})^2 = \frac{\partial n_i}{\partial x_i}\frac{\partial n_j}{\partial x_j}$$

$$(\hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}})^2 = n_i \epsilon_{ijk} \frac{\partial n_k}{\partial x_j} n_l \epsilon_{lmn} \frac{\partial n_n}{\partial x_m}$$

$$|\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}|^2 = \epsilon_{ijk} n_j \epsilon_{klm} \frac{\partial n_m}{\partial x_l} \epsilon_{inp} n_n \epsilon_{pqr} \frac{\partial n_r}{\partial x_q}$$

(b) For all subsequent parts of the question, we take the Frank constants to be equal, $K_1 = K_2 = K_3 = K$. Use index manipulations to show the energy may be simplified to: [25%]

$$E_f[\hat{\mathbf{n}}] = \int \frac{1}{2}K \left(|\nabla \cdot \hat{\mathbf{n}}|^2 + |\nabla \times \hat{\mathbf{n}}|^2\right) dV.$$

Applying the epsilon-delta identity to the to $\epsilon_{ijk}\epsilon_{inp}$ in K_3 term gives

$$|\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}|^2$$
$$= \epsilon_{ijk} n_j \epsilon_{klm} \frac{\partial n_m}{\partial x_l} \epsilon_{inp} n_n \epsilon_{pqr} \frac{\partial n_r}{\partial x_q}$$
$$= (\delta_{jn}\delta_{kp} - \delta_{jp}\delta_{kn}) n_j \epsilon_{klm} \frac{\partial n_m}{\partial x_l} n_n \epsilon_{pqr} \frac{\partial n_r}{\partial x_q}$$
$$= n_n \epsilon_{plm} \frac{\partial n_m}{\partial x_l} n_n \epsilon_{pqr} \frac{\partial n_r}{\partial x_q} - n_p \epsilon_{nlm} \frac{\partial n_m}{\partial x_l} n_n \epsilon_{pqr} \frac{\partial n_r}{\partial x_q}$$

 (TURN OVER

Since $\hat{\mathbf{n}}$ *is a unit vector,* $n_n n_n = 1$, *so this simplifies to*

$$
\begin{aligned}
=&\epsilon_{plm}\frac{\partial n_m}{\partial x_l}\epsilon_{pqr}\frac{\partial n_r}{\partial x_q} - n_p\epsilon_{nlm}\frac{\partial n_m}{\partial x_l}n_n\epsilon_{pqr}\frac{\partial n_r}{\partial x_q}\\
=&\epsilon_{plm}\frac{\partial n_m}{\partial x_l}\epsilon_{pqr}\frac{\partial n_r}{\partial x_q} - n_n\epsilon_{nlm}\frac{\partial n_m}{\partial x_l}n_p\epsilon_{pqr}\frac{\partial n_r}{\partial x_q}\\
=&|\nabla\times\hat{\mathbf{n}}|^2 - (\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}})^2
\end{aligned}
$$

proving the desired result.

(c) A drop of liquid crystal will adopt an alignment $\hat{\mathbf{n}}(\mathbf{x})$ that minimizes the Frank elastic energy, subject to the constraint that $\hat{\mathbf{n}}$ is a unit vector. By considering a directional derivative with respect to $\hat{\mathbf{n}}$ in the direction of $\delta\hat{\mathbf{n}}$, show that the minimizing alignment obeys the partial differential equation

$$\nabla^2\hat{\mathbf{n}} = \lambda(\mathbf{x})\hat{\mathbf{n}},$$

where $\lambda(\mathbf{x})$ is an unknown scalar field. You do not need to find $\lambda(\mathbf{x})$. [40%]

We introduce a Lagrange multiplier field $\lambda(\mathbf{x})$ *to implement the constraint* $n_i n_i = 1$. *Using index notation, the modified functional to be minimized has has the form*

$$E(\hat{\mathbf{n}}) = \int \frac{1}{2}K\left(\frac{\partial n_i}{\partial x_i}\frac{\partial n_j}{\partial x_j} + \epsilon_{ijk}\frac{\partial n_k}{\partial x_j}\epsilon_{imn}\frac{\partial n_n}{\partial x_m}\right) + \lambda(n_i n_i - 1)\mathrm{d}V.$$

Taking the directional derivative w.r.t. $\hat{\mathbf{n}}$ *in the direction of* $\delta\hat{\mathbf{n}}$ *given*

$$
\begin{aligned}
DE(\hat{\mathbf{n}})[\delta\hat{\mathbf{n}}] = &\frac{d}{d\epsilon}\left[\int \frac{1}{2}K\left(\frac{\partial(n_i+\epsilon\delta n_i)}{\partial x_i}\frac{\partial(n_j+\epsilon\delta n_j)}{\partial x_j} + \epsilon_{ijk}\frac{\partial(n_k+\epsilon\delta n_k)}{\partial x_j}\epsilon_{imn}\frac{\partial(n_n++\epsilon\delta n_n)}{\partial x_m}\right)\right.\\
&+\lambda((n_i+\epsilon\delta n_i)(n_i+\epsilon\delta n_i)-1)\mathrm{d}V]_{\epsilon=0}\,.\\
=&\int\frac{1}{2}K\left(2\frac{\partial\delta n_i}{\partial x_i}\frac{\partial n_j}{\partial x_j} + 2\epsilon_{ijk}\frac{\partial\delta n_k}{\partial x_j}\epsilon_{imn}\frac{\partial n_n}{\partial x_m}\right) + 2\lambda n_i\delta n_i\mathrm{d}V
\end{aligned}
$$

We swap k *and* i *indices in the curl term, so everything has a* δn_i, *then apply integration by parts using the divergence theorem:*

$$
\begin{aligned}
DE(\hat{\mathbf{n}})[\delta\hat{\mathbf{n}}] =& \int\frac{1}{2}K\left(2\frac{\partial\delta n_i}{\partial x_i}\frac{\partial n_j}{\partial x_j} + 2\epsilon_{kji}\frac{\partial\delta n_i}{\partial x_j}\epsilon_{kmn}\frac{\partial n_n}{\partial x_m}\right) + 2\lambda n_i\delta n_i\mathrm{d}V\\
=&\int K\left(-\delta n_i\frac{\partial}{\partial x_i}\frac{\partial n_j}{\partial x_j} - \epsilon_{kji}\delta n_i\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial n_n}{\partial x_m}\right) + 2\lambda n_i\delta n_i\mathrm{d}V\\
&+\text{surface term}\\
=&\int\left(K\left(-\frac{\partial}{\partial x_i}\frac{\partial n_j}{\partial x_j} - \epsilon_{kji}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial n_n}{\partial x_m}\right) + 2\lambda n_i\right)\delta n_i\mathrm{d}V + \text{surface term}
\end{aligned}
$$

(cont.

Since δn_i is arbitrary, we get the following vector pde

$$K\left(-\frac{\partial}{\partial x_i}\frac{\partial n_j}{\partial x_j} - \epsilon_{kji}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial n_n}{\partial x_m}\right) + 2\lambda n_i = 0$$

Applying the epsilon-delta identity then gives

$$K\left(-\frac{\partial}{\partial x_i}\frac{\partial n_j}{\partial x_j} - \frac{\partial}{\partial x_m}\frac{\partial n_i}{\partial x_m} + \frac{\partial}{\partial x_j}\frac{\partial n_j}{\partial x_i}\right) + 2\lambda n_i = 0$$

$$-K\frac{\partial}{\partial x_m}\frac{\partial n_i}{\partial x_m} + 2\lambda n_i = 0$$

$$K\nabla^2\hat{\mathbf{n}} = 2\lambda\hat{\mathbf{n}}$$

(d) The boundary of the drop has outward unit normal $\hat{\mathbf{m}}$. Find the energy minimizing boundary condition on $\hat{\mathbf{n}}$. Express your answer in index notation, and without permutation symbols. [20%]

The integration by parts in the previous question yields the surface term:

$$0 = \int \frac{1}{2}K\left(2\delta n_i\frac{\partial n_j}{\partial x_j}m_i + 2\epsilon_{kji}\delta n_i\epsilon_{kmn}\frac{\partial n_n}{\partial x_m}m_j\right)\mathrm{d}S$$

Again, since δn_i is arbitrary, this yields the minimizing condition:

$$0_i = \frac{\partial n_j}{\partial x_j}m_i + \epsilon_{kji}\epsilon_{kmn}\frac{\partial n_n}{\partial x_m}m_j$$

Applying the epsilon-delta identity, we get as the final boundary condition:

$$0_i = \frac{\partial n_j}{\partial x_j}m_i + \frac{\partial n_i}{\partial x_j}m_j - \frac{\partial n_j}{\partial x_i}m_j.$$

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