Version PAD/4

EGT3/EGT2

ENGINEERING TRIPOS PART IIB

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Friday 29 April 2022 2 to 3.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS

Write on single-sided paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationary from the Examination Room.

1 (a) State the definition of the group velocity in three dimensions and name three important properties of the group velocity so defined. [10%]

(b) Internal waves in a rotating, stratified ocean are governed by the wave-like equation

$$\frac{\partial^2}{\partial t^2}\nabla^2 u_z + (2\Omega)^2 \frac{\partial^2 u_z}{\partial z^2} + N^2\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) = 0 \ ,$$

where u_z is the vertical velocity, z the vertical coordinate, Ω the rotation rate, and the constant N is a measure of the strength of the stratification.

(i) Show that the dispersion relationship takes the form

$$\varpi^2 = N^2 + f(N,\Omega)\frac{k_z^2}{k^2} \ ,$$

where $k = |\mathbf{k}|$. Find the function $f(N, \Omega)$. [15%]

(ii) Show that the group velocity takes the form

$$\mathbf{c}_g = g(N,\Omega)\frac{k_\perp^2 \mathbf{k}_{//} - k_z^2 \mathbf{k}_\perp}{\varpi k^4} \ ,$$

where $\mathbf{k}_{//} = k_z \hat{\mathbf{e}}_z$, $\mathbf{k}_\perp = \mathbf{k} - \mathbf{k}_{//}$ and $k_\perp = |\mathbf{k}_\perp|$. Find the function g. [30%]

(iii) Use the governing equation to explain why the group velocity is zero when $N = 2\Omega$. [15%]

(c) In well mixed regions of the oceans we may take $N = 0$.

(i) In such cases, show that the group velocity takes the form

$$\mathbf{c}_g = \pm 2\Omega G(k_x, k_y, k_z),$$

and find the function $G(k_x, k_y, k_z)$. [20%]

(ii) A wave generator of fixed frequency is placed in a well-mixed region of the ocean. Sketch the dispersion pattern for $\varpi \ll \Omega$ and $\varpi = 3\Omega$, explaining why the patterns take the forms they do. [10%]

2 (a) Under what circumstances is a self-similar solution of the diffusion equation expected? Use dimensional analysis to show that, under those conditions, the diffusion length takes the form $\ell \sim \sqrt{\alpha t}$, where α is the diffusivity. [15%]

(b) A small radioactive heat source is placed at the origin in a thermally conducting medium of infinite extent. The temperature far from the source is $T = 0$ and heat diffuses through the medium to give a spherically symmetric temperature field, governed by

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \ .$$

(i) If r is the distance from the source, show that the related field $\vartheta = rT(r,t)$ is governed by

$$\frac{\partial \vartheta}{\partial t} = \alpha \frac{\partial^2 \vartheta}{\partial r^2} \ .$$ [10%]

(ii) The field ϑ takes the self-similar form $\vartheta = Af\left(r/\left(2\sqrt{\alpha t}\right)\right) = Af(\eta)$, where A is a dimensional constant and $f(0) = 1$. Show that f is governed by

$$f''(\eta) + 2\eta f'(\eta) = 0 \ .$$ [20%]

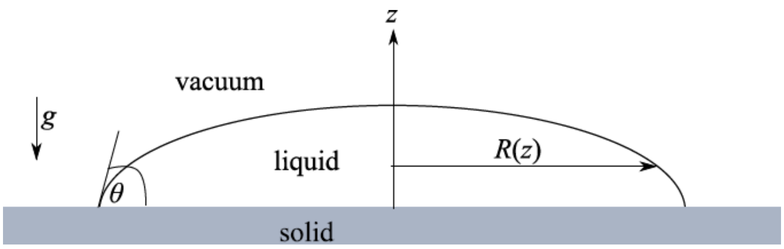
(iii) Find the function $f(\eta)$ and hence the temperature field

$$T(r,t) = \frac{A}{r} f(\eta) \, .$$ [20%]

(iv) Fourier's law of heat conduction says that the heat flux per unit area is given by $\dot{q} = -k\left(dT/dr\right)$, where k is the thermal conductivity. If $\dot{Q}$ is the *net* heat flux from the radioactive source, find the relationship between A and $\dot{Q}$. [10%]

(c) A steady temperature field is maintained in a conducting medium of finite extent by specifying the temperature distribution on the surface of the medium, say $T = T_{\text{surf}}(\mathbf{x})$. Prove that the resulting temperature field is unique. [25%]

3 An axisymmetric drop of liquid of density ρ sits on a planar surface in a gravitational field g, as shown below.



The system contains three interfaces, each of which costs an interfacial energy γ_i per unit area, with i = 1 being liquid-vacuum, i = 2 liquid-solid and i = 3 solid-vacuum. Thus, if each interface has an area A_i, the total interfacial energy is

$$E_{\text{int}} = A_1\gamma_1 + A_2\gamma_2 + A_3\gamma_3 .$$

The drop's shape minimises its gravitational and interfacial energy for a fixed volume.

(a) The shape is described by the function $R(z)$, with z = 0 being the solid and z = h being the top of the drop. $R(z)$ minimises a functional of the form

$$E(R) = \int_0^h I(R, R'(z), z)\mathrm{d}z + AR(0)^2 + B .$$

Find expressions for the integrand, $I(R, R', z)$, and the constant A. You do not need to find the constant B. [20%]

(b) The drop makes a contact angle θ with the solid, as marked on the diagram. Use the directional derivative of $E(R)$ to find $\cos(\theta)$. [30%]

(c) Assume gravity is negligible.

(i) Show that $R(z)$ obeys the differential equation

$$\lambda R = \frac{1}{\sqrt{1+R'^2}}$$

where λ is a constant. [30%]

(ii) Find the functional form of $R(z)$, and give a geometric interpretation. You do not need to determine the value of any constants in your solution. [20%]

[Hint: You may regard h as a known quantity, so you do not need to minimise over h.]

4 A liquid crystal is a fluid of rod shaped molecules, in which the molecules align locally along a direction described by a vector $\hat{\mathbf{n}}$ of unit length. If the alignment direction $\hat{\mathbf{n}}(\mathbf{x})$ is spatially varying then the liquid crystal has a Frank elastic energy,

$$E_F(\hat{\mathbf{n}}) = \int \left[\tfrac{1}{2} K_1 (\nabla \cdot \hat{\mathbf{n}})^2 + \tfrac{1}{2} K_2 (\hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}})^2 + \tfrac{1}{2} K_3 \, |\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}|^2 \right] dV \ ,$$

where K_1, K_2 and K_3 are known as the Frank constants, and the integral is over the volume of the liquid crystal.

(a) Express the integrand of the Frank energy using index notation and the summation convention. [15%]

(b) For all subsequent parts of this question, we take the Frank constants to be equal, $K_1 = K_2 = K_3 = K$. Use index manipulations to show the energy may be simplified to:

$$E_F(\hat{\mathbf{n}}) = \int \left[\tfrac{1}{2} K \left((\nabla \cdot \hat{\mathbf{n}})^2 + |\nabla \times \hat{\mathbf{n}}|^2 \right) \right] dV \ . \qquad [25\%]$$

(c) A drop of liquid crystal will adopt an alignment $\hat{\mathbf{n}}(\mathbf{x})$ that minimises the Frank elastic energy, subject to the constraint that $\hat{\mathbf{n}}$ is a unit vector. By considering a directional derivative with respect to $\hat{\mathbf{n}}$ in the direction of $\delta\hat{\mathbf{n}}$, show that the minimising alignment obeys the partial differential equation

$$\nabla^2 \hat{\mathbf{n}} = \lambda(\mathbf{x}) \hat{\mathbf{n}} \, ,$$

where $\lambda(\mathbf{x})$ is an unknown scalar field. You do not need to find $\lambda(\mathbf{x})$. [40%]

(d) The boundary of the drop has outward unit normal $\hat{\mathbf{m}}$. Find the energy-minimizing boundary condition on $\hat{\mathbf{n}}$. Express your answer in index notation, and without permutation symbols. [20%]

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