Crib

4M12 2023, JSB/1

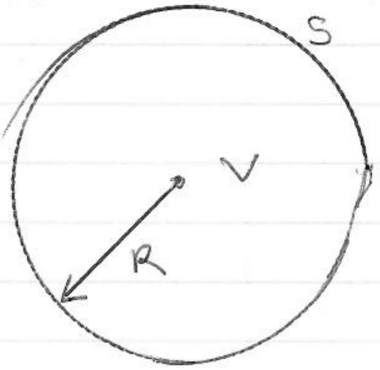
Q1

(a) $\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{-1}{4\pi r}\right) = +\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r} r^2\left(\frac{1}{r^2}\right) = 0$

$(r \neq 0)$

$$\int_V \nabla^2\Phi\, dv = \oint_S (\nabla\Phi)\cdot d\underline{S} = \oint_S \frac{\hat{R}_r}{4\pi R^2}\cdot d\underline{S}$$

$$\Rightarrow \int_V \nabla^2\Phi\, dv = \frac{1}{4\pi R^2}\oint dS = 1$$



(b) If δ-function is located at $\underline{x}'$ and has strength $S(\underline{x}')$, then the soln in (a) becomes

$$\Phi = -\frac{S(\underline{x}')}{4\pi|\underline{x}-\underline{x}'|}$$

For a distributed source $S(x)$, superposition gives

$$\Phi(\underline{x}) = -\frac{1}{4\pi}\int \frac{S(\underline{x}')}{|\underline{x}-\underline{x}'|}\, dV'$$

(c) $$\nabla^2 \underline{A} = -\mu_0 \underline{J}(x)$$

Can apply the result of (b) one component at a time, so

$$\underline{A}(x) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, dV'$$

$$\underline{B} = \nabla\times\underline{A} = \frac{\mu_0}{4\pi}\int \nabla\times\left[\frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\right] dV'$$

↑ opperates on $\underline{x}$ keeping $\underline{x}'$ constant

$$\nabla\times\left[\frac{\underline{J}'}{|\underline{x}-\underline{x}'|}\right] = \frac{1}{|\underline{x}-\underline{x}'|}\underbrace{\nabla\times\underline{J}'}_{0} + \nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)\times\underline{J}(\underline{x}')$$

$$= -\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}\times\underline{J}(\underline{x}')$$

$$\Rightarrow \underline{B}(\underline{x}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{x}')\times\underline{r}}{r^3}\, dV' \quad , \quad \underline{r} = \underline{x}-\underline{x}'$$

(d) $$\underset{\sim}{B}(\underset{\sim}{x},t) = \frac{\mu_0}{4\pi}\int \frac{\underset{\sim}{J}(\underset{\sim}{x}',t)\times\underset{\sim}{r}}{|\underset{\sim}{r}|^3}\,dV' \,, \quad \underset{\sim}{r} = \underset{\sim}{x}-\underset{\sim}{x}'$$

cannot be correct because changes in $\underset{\sim}{J}(\underset{\sim}{x}')$ take a finite time, $|\underset{\sim}{r}|/c$ (c = speed of light), to be felt at $\underset{\sim}{x}$. To correct for this we write

$$\underset{\sim}{A}(\underset{\sim}{x},t) = \frac{\mu_0}{4\pi}\int \frac{\underset{\sim}{J}(\underset{\sim}{x}', t-\frac{r}{c})}{|\underset{\sim}{r}|}\,dV'$$

where $t-|\underset{\sim}{r}|/c$ allows for the finite time of flight from $\underset{\sim}{x}$ to $\underset{\sim}{x}'$. Thus

$$\underline{\underline{\underset{\sim}{B}(\underset{\sim}{x},t) = \nabla\times\left(\underset{\sim}{A}(\underset{\sim}{x},t)\right) = \frac{\mu_0}{4\pi}\int \nabla\times\left[\frac{\underset{\sim}{J}(\underset{\sim}{x}', t-\frac{r}{c})}{|\underset{\sim}{r}|}\right]dV'}}$$

Q2

4M12 2023 CRIB

(a) (i) Need to write $k = \partial\theta/\partial x$, $\varpi = -\frac{\partial\theta}{\partial t}$

in order to recover local form $\eta \sim A\exp[i(kx-\varpi t)]$

Thus, $$\frac{\partial k}{\partial t} = \frac{\partial^2\theta}{\partial x\partial t} = -\frac{\partial\varpi}{\partial x} = -\frac{d\varpi}{dk}\frac{\partial k}{\partial x}$$

But $c_g = \frac{d\varpi}{dk}$, so $\underline{\underline{\frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0}}$

(ii) write $k = f(x - c_g t) = f(X)$

$$\frac{\partial k}{\partial t} = -f'(X)\frac{\partial}{\partial t}(c_g t) = -f'(X)\left[c_g + \frac{dc_g}{dk}\frac{\partial k}{\partial t}t\right]$$

$$\Rightarrow \underline{\underline{\left[1 + f'(X)\frac{dc_g}{dk}t\right]\frac{\partial k}{\partial t} = -f'(X)c_g}} \qquad \text{———①}$$

$$\frac{\partial k}{\partial x} = f'(X)\left[1 - \frac{dc_g}{dk}\frac{\partial k}{\partial x}t\right]$$

$$\Rightarrow \underline{\underline{\left[1 + f'(X)\frac{dc_g}{dk}t\right]\frac{\partial k}{\partial x} = f'(X)}} \qquad \text{———②}$$

Compare ① and ② $\underline{\underline{\frac{\partial k}{\partial t} = -c_g\frac{\partial k}{\partial x}}}$, as required.

Since $k = f(x - c_g(k)t)$, then k is constant if $x - c_g(k)t = \text{const}$. Thus k is constant along trajectories $\frac{dx}{dt} = c_g$.

$\Rightarrow$ Need to travel at speed c_g to keep seeing waves of wavenumber k.

(b) (i) Dispersion relationship is

$$G k^4 + \kappa^4 G = \rho\varpi^2$$

or

$$\underline{\underline{\varpi^2 = \frac{G}{\rho}(k^4 + \kappa^4)}}$$

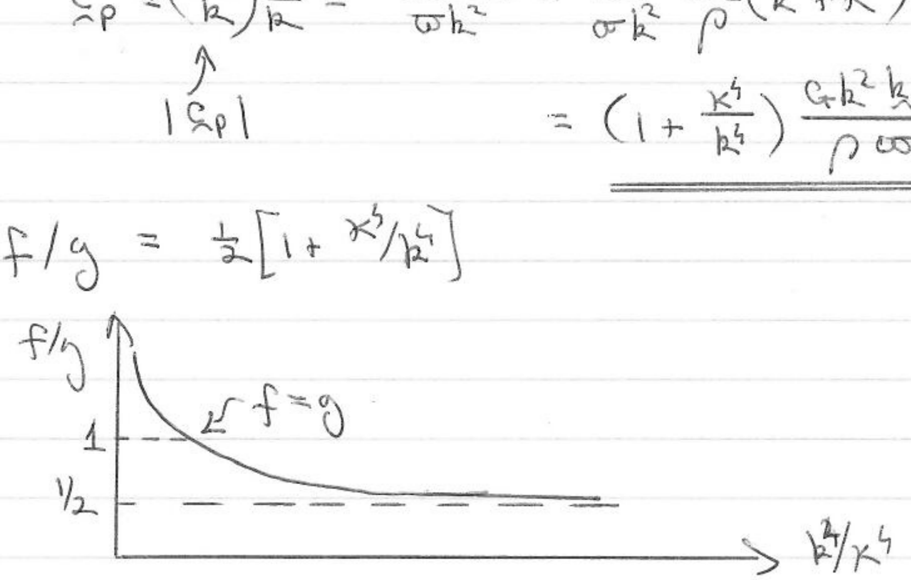
$$\Rightarrow \quad 2\varpi \underset{\sim}{c_g} = \frac{G}{\rho}4k^2\underset{\sim}{k} \quad \Rightarrow \quad \underline{\underline{\underset{\sim}{c_g} = \frac{2Gk^2\underset{\sim}{k}}{\rho\varpi}}} \quad \Rightarrow \underline{\underline{g = 2}}$$

(b)(i) cont.

$$\underset{\sim}{c}_p = \underset{\substack{\uparrow \\ |\underset{\sim}{c}_p|}}{\left(\frac{\omega}{k}\right)}\frac{\underset{\sim}{k}}{k} = \frac{\omega^2 \underset{\sim}{k}}{\omega k^2} = \frac{\underset{\sim}{k}}{\omega k^2}\frac{G}{\rho}\left(k^4+\kappa^4\right)$$

$$= \underline{\underline{\left(1+\frac{\kappa^4}{k^4}\right)\frac{G k^2 \underset{\sim}{k}}{\rho\,\omega}}} \qquad \underline{\underline{f = 1+\frac{\kappa^4}{k^4}}}$$

(ii) $f/g = \frac{1}{2}\left[1+\kappa^4/k^4\right]$

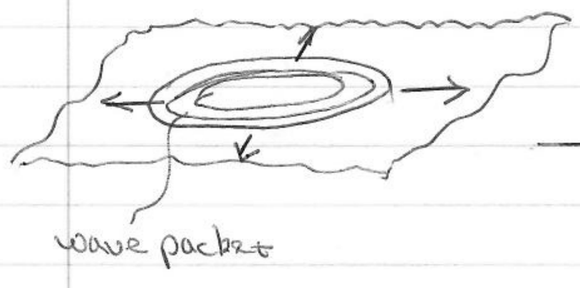


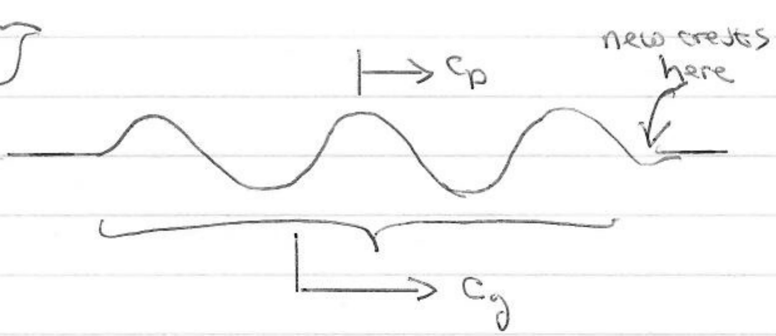
There is only one k^4/κ^4 that ensures $f=g$

By inspection, this is $\underline{\underline{|\underset{\sim}{k}| = \kappa}}$.

(iii) $|\underset{\sim}{k}| \sim \ell^{-1}$ so $\kappa\ell \ll 1 \Rightarrow \frac{\kappa}{|\underset{\sim}{k}|} \ll 1 \Rightarrow \frac{|\underset{\sim}{k}|}{\kappa} \gg 1$

From (b)(ii) above, $|\underset{\sim}{k}| \gg \kappa \Rightarrow f<g \Rightarrow \underline{\underline{|\underset{\sim}{c}_p| < |\underset{\sim}{c}_g|}}$





Since $c_g > c_p$ new crests continully appear at the front of the wave packet, ripple through the packet, and dissapear at the rear of the packet.

3 A cylindrical optical fiber has a refractive index, $n(r)$, that varies with radius. Using standard cylindrical coordinates, a light ray through the fiber follows a path $(r(z), \theta(z), z)$.

(a) Find a functional $T(r, \theta)$ for the time taken for a ray to travel from $z = A$ to $z = B$. [10%]

Length element in cylindrical coordinates is $\sqrt{dz^2 + dr^2 + r^2 d\theta^2}$.

Speed=distance/time, so integrating and pulling dz out of the square-root gives

$$T = \int_A^B \frac{1}{c} n(r)\sqrt{1 + r'^2 + r^2\theta'^2}\ \mathrm{d}z \equiv \int_A^B L(r, r', \theta')\mathrm{d}z.$$

(b) According to Fermat's principle, rays will follow paths that minimize T. Show that Fermat's principal leads to the following two equations [30%]

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(r(z)^2\theta'(z)\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{n(r)}{\sqrt{r'(z)^2 + r(z)^2\theta'(z)^2 + 1}}\right) = 0.$$

[Hint: Consider how the Beltrami special-case of the Euler-Lagrange equation works for a functional of two functions.]

We are minimizing T with respect to two functions, $\theta(z)$ *and* $r(z)$.

The integrand $L(r, r', \theta')$ *does not depend on* θ*, so the standard E-L equation for* $\theta(z)$ *simply reads*

$$\frac{d}{dz}\frac{\partial L}{\partial \theta'} = 0 \ \rightarrow\ \frac{d}{dz}\left(\frac{n(r)r^2\theta'}{\sqrt{1 + r'^2 + r^2\theta'^2}}\right) = 0.$$

The integrand does depend on r, so this EL-equation is not so simple. However, L does not explicitly depend on z, so we may use the Beltrami manipulated form

$$\frac{d}{dz}\left(L - r'\frac{\partial L}{\partial r'} - \theta'\frac{\partial L}{\partial \theta'}\right) = 0$$

$$\frac{d}{dz}\left(n(r)\sqrt{1 + r'^2 + r^2\theta'^2} - \frac{n(r)r'^2}{\sqrt{1 + r'^2 + r^2\theta'^2}} - \frac{n(r)r^2\theta'^2}{\sqrt{1 + r'^2 + r^2\theta'^2}}\right) = 0$$

$$\frac{d}{dz}\left(\frac{n(r)}{\sqrt{1 + r'^2 + r^2\theta'^2}}\right) = 0$$

This is the second required equation. Combining it with the θ *E-L equation gives* $\frac{d}{dz}\left(r^2\theta'\right) = 0$, *as required for the first.*

Length element in cylindrical coordinates is $\sqrt{dz^2 + dr^2 + r^2 d\theta^2}$.

Speed=distance/time, so integrating and pulling dz out of the square-root gives

$$T = \frac{1}{c}\int_A^B n(r)\sqrt{1 + r'^2 + r^2\theta'^2}\ \mathrm{d}z$$

(c) What may be concluded about a ray that enters the fiber in an $r - z$ plane [10%]

Integrating the two equations above, we get

$$r(z)^2\theta'(z) = c, \qquad \frac{n(r)}{\sqrt{r'(z)^2 + r(z)^2\theta'(z)^2 + 1}} = d.$$

where c and d are constants of integration. In this case $\theta' = 0$ on entry, so $c = 0$, indicating that θ' must remain zero throughout — i.e. the ray remains confined to this $r - z$ plane.

(d) Find the refractive index profile $n(r)$ that allows a helical ray at any radius. Set the constants of integration such that the helix at radius r_0 progresses by $\Delta z = p$ in each turn. [20%]

A helix has constant $r(z) = r_0$, and constant $\theta'(z) = 2\pi/p$.

Substituting for θ' in the second equation of motion from the first gives:

$$\frac{n(r)}{\sqrt{r'^2 + c^2/r^2 + 1}} = d.$$

For $r' = 0$ to be a solution, we need

$$n(r) = d\sqrt{1 + c^2/r^2}.$$

The first equation of motion gives the value of c as

$$r_0^2 2\pi/p = c$$

So we have

$$n(r) = d\sqrt{1 + 4\pi^2 r_0^4/(p^2r^2)}.$$

The second constant, d, just scales the overall velocity, so it has no impact on the form of the minimum time paths.

(e) The profile is actually $n(r) = n_0/(1 + \lambda r)$. Show a ray that enters directed in an $r - z$ plane will move in a circle. [30%]

The equation of motion simply gives $\theta' = 0$, i.e. the ray remains in the $r - z$ plane.

The second equation of motion becomes

$$d_2 = (1 + \lambda r)^2 \left(r'(z)^2 + 1\right)$$

where we have redefined the arbitrary constant $(n_0/d)^2 = d_2$. Rearranging for r',

$$\frac{dr}{dz} = \sqrt{\frac{d_2}{(1 + \lambda r)^2} - 1} = \frac{\sqrt{d_2 - (1 + \lambda r)^2}}{(1 + \lambda r)}.$$

Dividing and integrating:

$$\int \frac{(1+\lambda r)}{\sqrt{d_2 - (1+\lambda r)^2}} \mathrm{d}r = \int \mathrm{d}z = z + d_3.$$

The r.h.s integral is straightforward as the numerator is the derivative of the denominator,

$$-\frac{1}{\lambda}\sqrt{d_2 - (1+\lambda r)^2} = z + d_3.$$

Squaring, we have

$$\frac{d_2}{\lambda^2} = (1/\lambda + r)^2 + (z + d_3)^2.$$

which is the equation of a circle.

4 A light, inextensible cantilever of length L and bending stiffness B is clamped on the left and loaded with a point weight mg on the right, resulting in a vertical downward deflection $y(x)$ as shown in Fig. 1. The elastic potential energy of the cantilever is proportional to its curvature squared which, for small deflections, we may take to be $E = \int_0^L \frac{1}{2} K y''(x)^2 \mathrm{d}x$.

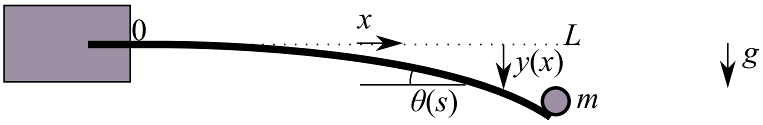


Fig. 1

(a) The cantilever deformation minimizes the total potential energy. We first seek an approximate solution using the Rayleigh-Ritz method with a trial function of the form:

$$y(x) = \sum_{i=0}^{N} a_i \left(\frac{x}{L}\right)^i .$$

(i) Explain why a_0 and a_1 must be set to zero before starting the procedure. [5%]

For Rayleigh-Ritz, the trial function must obey the fixed (clamped) boundary conditions - in this case $y(0) = y'(0) = 0$, *hence* $a_0 = a_1 = 0$.

(ii) Find the Rayleigh-Ritz approximate solution for $N = 2$. [15%]

For $N = 2$, *we have* $y(x) = a_2(x/L)^2$. *Substituting this into the total energy*

$$E = \int_0^L 2Ka_2^2/L^4 \mathrm{d}x - mga_2 = 2Ka_2^2/L^3 - mga_2.$$

Minimizing w.r.t a_2 *gives*

$$mg = 4Ka_2/L^3 \quad \rightarrow \quad a_2 = mgL^3/(4K).$$

$N = 2$ *solution is*

$$y(x) = \frac{mgL}{4K} x^2.$$

(b) Use a directional derivative of the potential energy to find a differential equation for $y(x)$, and the appropiate boundary conditions. [25%]

Total potential energy is

$$E = \int_0^L \frac{1}{2} K y''(x)^2 \mathrm{d}x - mgy(L).$$

Directional derivative w.r.t y *in direction* δy *gives*

$$DE(y)\delta y = \int_0^L K y'' \delta y'' \mathrm{d}x - mg\delta y(L).$$

(cont.

Integrating by parts twice gives

$$DE(y)\delta y = \int_0^L Ky''''\delta y \mathrm{d}x + [Ky''\delta y']_0^L - [Ky'''\delta y]_0^L - mg\delta y(L).$$

At the $x = 0$ *boundary we have clamped conditions, so* $\delta y(0) = \delta y'(0) = 0$*, so these terms vanish in the directional derivative.*

$$DE(y)\delta y = \int_0^L Ky''''\delta y \mathrm{d}x + Ky''(L)\delta y'(L)] - Ky'''(L)\delta y(L) - mg\delta y(L).$$

For minimum energy we need the derivative to vanish for all deltay. This gives the equation

$$Ky''''(x) = 0$$

And the boundary conditions $y''(L) = 0$*,* $y'''(L) = -mg/K$*. At* $x = 0$ *we have clamped conditions* $y(0) = y'(0) = 0$*.*

(c) Find the analytic solution $y(x)$, and discuss its relation to Rayleigh-Ritz solutions for $N = 2, 3, 4$. [15%]

The equation integrates four times to give

$$y(x) = a_0 + a_1\frac{x}{L} + a_2\frac{x^2}{L^2} + a_3\frac{x^3}{L^3}$$

Boundary conditions at $x = 0$ *give* $a_0 = a_1 = 0$*.*

Boundary conditions at $x = L$ *give*

$$2a_2 + 6a_3 = 0$$

$$\frac{6a_3}{L^3} = -mg/K \rightarrow a_3 = \frac{-mgL^3}{6K}$$

Substituting above then gives $a_2 = \frac{mgL^3}{2}$*, and hence the form of the beam is*

$$y(x) = mgL^3(\frac{x^2}{2L^2} - \frac{x^3}{6L^3}) = \frac{mgx^2}{6K}(3L - x).$$

For $N = 3$ *and* $N = 4$ *the RR trial function includes the exact answer, so it will give the exact result. For* $N = 2$ *this is not true, so the RR will be an approximate solution with higher energy.*

(d) Alternativley, the deflection may be described by $\theta(s)$ the angle between the cantilever and the horizontal as a function of arc-length, as shown in Fig. 1. The elastic energy is now $E = \int_0^L \frac{1}{2}K\theta'(s)^2\mathrm{d}s$, and there is no assumption about small deflections.

(i) Find the differential equation for $\theta(s)$, and the appropiate boundary conditions. [20%]

The vertical displacement of the end is now $y(L) = \int_0^L \sin(\theta)\mathrm{d}s$*, so the potential energy is*

$$E = \int_0^L \frac{1}{2}K\theta'(s)^2 - mg\sin(\theta(s))\mathrm{d}s$$

Taking the directional derivative with respect to θ *in the direction* $\delta\theta$ *gives*

$$DE(\theta)\delta\theta = \int_0^L K\theta'(s)\delta\theta' - mg\cos(\theta)\delta\theta\mathrm{d}s.$$

Integrating by parts once,

$$DE(\theta)\delta\theta = \int_0^L -K\theta''\delta\theta - mg\cos(\theta)\delta\theta\mathrm{d}s + [K\theta'\delta\theta]_0^L.$$

At $s = 0$ *we have* $\theta = 0$ *and hence* $\delta\theta = 0$*, so this gives*

$$DE(\theta)\delta\theta = \int_0^L -K\theta''\delta\theta - mg\cos(\theta)\delta\theta\mathrm{d}s + K\theta'(L)\delta\theta(L).$$

Setting this to zero for all permitted $\delta\theta$ *gives*

$$K\theta'' + mg\cos(\theta) = 0$$

with the boundary conditions $\theta(0) = 0$ *and* $\theta'(L) = 0$*.*

(ii) The cantilever deflects to a final angle $\theta(L) = \theta_f$. Find the value of $\theta'(0)$. [20%]

The integrand of the energy does not depend on s, so we may use the Beltrami special form

$$\frac{d}{ds}\left(\frac{1}{2}K\theta'^2 - mg\sin(\theta) - \theta'K\theta'\right) = 0$$

which we can integrate to get

$$\frac{1}{2}K\theta'^2 + mg\sin(\theta) = c = mg\sin(\theta_f).$$

where the constant of integration has been fixed using the data at the r.h.s. At the lhs, we thus have

$$\theta'(0) = \frac{2mg\sin(\theta_f)}{K}.$$

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