Version JSB/3

EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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Friday 28 April 2023 2.00 to 3.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 (a) Consider the Poisson equation

$$\nabla^2\Phi = \delta(\mathbf{x}),$$

where $\delta(\mathbf{x})$ is the three-dimensional delta function, defined by

$$\delta(\mathbf{x}) = 0 \text{ for } \mathbf{x} \neq \mathbf{0}; \qquad \int_V \delta(\mathbf{x})\mathrm{d}V = 1,$$

and V is a spherical volume centered on the origin. Show that

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}$$

is the solution to this Poisson equation, in the sense that

$$\nabla^2\Phi = 0 \text{ for } \mathbf{x} \neq \mathbf{0}; \qquad \int_V \nabla^2\Phi\mathrm{d}V = 1.$$

[25%]

(b) Use superposition, plus the results of part (a) above, to deduce the solution to the Poisson equation

$$\nabla^2\Phi = S(\mathbf{x}),$$

where $S(\mathbf{x})$ is a general source term. [20%]

(c) A static magnetic field, $\mathbf{B}$, has a vector potential, $\mathbf{A}$, defined by $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. This vector potential is related to the current density vector, $\mathbf{J}(\mathbf{x})$, by

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}(\mathbf{x}),$$

where μ_0 is the permeability of free space. Use the results of part (b) above to show that, in an infinite domain,

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \nabla \times \left[\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right]\mathrm{d}V',$$

where ∇ operates on $\mathbf{x}$ while treating $\mathbf{x}'$ as constant. Hence deduce the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}V', \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}'.$$

[35%]

(d) For a quasi-static magnetic field, $\mathbf{B}(\mathbf{x}, t)$, which varies slowly with time, the Biot-Savart law is usually written as,

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}', t) \times \mathbf{r}}{|\mathbf{r}|^3} \mathrm{d}V', \qquad \mathbf{r} = \mathbf{x} - \mathbf{x}'.$$

Explain why, given the finite speed of light, this cannot be strictly correct. How would you modify the quasi-static equation

$$\mathbf{B}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \nabla \times \left[\frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \right] \mathrm{d}V',$$

to allow for the finite speed of light? Explain your physical reasoning. [20%]

2 (a) A slowly modulated wave train takes the form

$$\eta(x,t) = A(x,t)\exp\left[i\theta(x,t)\right],$$

where A is the local amplitude and θ the phase function. At any one location a good approximation to the wave train is

$$\eta(x,t) = A\exp\left[i(kx - \varpi t)\right],$$

where $k(x,t)$ and $\varpi(x,t)$ are the local values of the wavenumber and angular frequency, which are constrained to satisfy the dispersion relationship $\varpi = \varpi(k)$.

(i) Express k and ϖ in terms of the phase function and hence show that k is governed by the simple equation

$$\frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0,$$

where $c_g(k)$ is the group velocity. [20%]

(ii) Show that the solution of this equation takes the functional form

$$k = h\left(x - c_g t\right), \quad \text{where } c_g \text{ itself is a function of } k,$$

and use this to deduce the fundamental property of group velocity for one-dimensional dispersive systems. [30%]

(b) A thin metal plate sits on an elastic foundation. The plate has a mass per unit area of ρ and a flexural rigidity of G. The stiffness of the foundation can be written as $\kappa^4 G$ and transverse vibrations of the plate are governed by

$$G\nabla^2\left(\nabla^2\eta\right) + \kappa^4 G\eta + \rho\frac{\partial^2\eta}{\partial t^2} = 0,$$

where η is the the lateral displacement of the plate.

(i) Show that the phase and group velocities can be written as

$$\mathbf{c}_p = f\left(k^4/\kappa^4\right)\frac{Gk^2\mathbf{k}}{\rho\varpi}, \qquad \mathbf{c}_g = g\left(k^4/\kappa^4\right)\frac{Gk^2\mathbf{k}}{\rho\varpi}$$

where $\mathbf{k}$ and ϖ are the wavevector and frequency, respectively, and f and g are functions of k^4/κ^4. Find the functions f and g. [20%]

(ii) Sketch the ratio f/g as a function of k^4/κ^4, and show that the phase and group velocities have equal magnitudes only when $|\mathbf{k}| = \kappa$. [10%]

(iii) The plate is struck by an object of scale ℓ where $\kappa\ell \ll 1$, and waves propagate radially outward from the point of impact. Sketch the form of the resulting wave pattern, showing the motion of the wave crests relative to the overall motion of the wave packet. [20%]

3 A cylindrical optical fiber has a refractive index, $n(r)$, that varies with radius. Expressed in standard cylindrical coordinates, a light ray through the fiber follows a path $(r(z), \theta(z), z)$.

(a) Find a functional $T(r, \theta)$ for the time taken for a ray to travel from $z = A$ to $z = B$. [10%]

(b) According to Fermat's principle, rays will follow paths that minimize T. Show that Fermat's principal leads to the following two equations [30%]

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(r(z)^2\theta'(z)\right) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{n(r)}{\sqrt{r'(z)^2 + r(z)^2\theta'(z)^2 + 1}}\right) = 0.$$

[Hint: Consider how the Beltrami special-case of the Euler-Lagrange equation works for a functional of two functions.]

(c) What may be concluded about a ray that enters the fiber in an $r - z$ plane. [10%]

(d) Find the refractive index profile $n(r)$ that allows a helical ray at any radius. Set the constants of integration such that the helix at radius r_0 progresses by $\Delta z = p$ in each turn. [20%]

(e) The profile is actually $n(r) = n_0/(1 + \lambda r)$. Show a ray that enters directed in an $r - z$ plane will move in a circle. [30%]

4 A light, inextensible cantilever of length L and bending stiffness B is clamped on the left and loaded with a point weight mg on the right, resulting in a vertical downward deflection $y(x)$ as shown in Fig. 1. The elastic potential energy of the cantilever is proportional to its curvature squared which, for small deflections, we may take to be $E = \int_0^L \frac{1}{2} K y''(x)^2 \mathrm{d}x$.

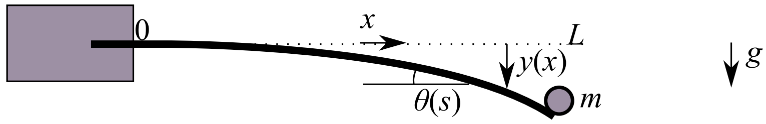

Fig. 1

(a) The cantilever deformation minimizes the total potential energy. We first seek an approximate solution using the Rayleigh-Ritz method with a trial function of the form:

$$y(x) = \sum_{i=0}^{N} a_i \left(\frac{x}{L}\right)^i .$$

(i) Explain why a_0 and a_1 must be set to zero before starting the procedure. [5%]

(ii) Find the Rayleigh-Ritz approximate solution for $N = 2$. [15%]

(b) Use a directional derivative of the potential energy to find a differential equation for $y(x)$, and the appropiate boundary conditions. [25%]

(c) Find the analytic solution $y(x)$, and discuss its relation to Rayleigh-Ritz solutions for $N = 2, 3, 4$. [15%]

(d) Alternativley, the deflection may be described by $\theta(s)$ the angle between the cantilever and the horizontal as a function of arc-length, as shown in Fig. 1. The elastic energy is now $E = \int_0^L \frac{1}{2} K \theta'(s)^2 \mathrm{d}s$, and there is no assumption about small deflections.

(i) Find the differential equation for $\theta(s)$, and the appropiate boundary conditions. [20%]

(ii) The cantilever deflects to a final angle $\theta(L) = \theta_f$. Find the value of $\theta'(0)$. [20%]

END OF PAPER