Version JSB/5

EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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Friday 26 April 2024 2 to 3.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 We consider the method of separation of variables to solve Laplace's equation.

(a) Explain when the method of separation of variables can be used. [10%]

(b) Consider Laplace's equation in the polar coordinate system

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \tag{1}$$

which governs a temperature distribution $f(r, \theta)$ in the disk $r \leq 2$ and $0 \leq \theta \leq 2\pi$. Assume a solution of the form $f = R(r)X(\theta)$. Use the method of separation of variables to deduce differential equations for R and X. [20%]

(c) After the separation, a constant k is obtained such that

$$F\left(R, dR/dr, d^2R/dr^2\right) = G\left(X, dX/d\theta, d^2X/d\theta^2\right) = k$$

Eliminate all invalid solutions and hence find the most general solution of Eqn. (1). [30%]

(d) Find the solution of Eqn. (1) with the boundary condition $f(2, \theta) = 2\cos(\theta) + \sin(2\theta)$. [20%]

(e) Assume the boundary condition at $r = 2$ is $f(2, \theta) = g(\theta)$, where g is an unknown function of θ. What condition must g satisfy? Explain how you would solve Eqn. (1) with this boundary condition. [20%]

2 A model equation for traffic flow assumes that the velocity of the cars is given by $u(x,t) = 1 - \rho(x,t)$, where $\rho(x,t)$ is the density of cars, x is position and t is time. Furthermore, $\rho(x,t)$ evolves according the differential equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho(1-\rho))}{\partial x} = 0$$

(a) Consider the initial condition

$$\rho(x,0) = \begin{cases} 1, & x > 1 \\ x, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

(i) Find $\rho(x,t)$ by integrating along characteristic lines. [20%]

(ii) Do discontinuities form in this case? If so, indicate where and when a discontinuity first occurs, and find the density distribution ρ at this instant. If discontinuities do not form, find the density distribution ρ at $t = 1$. [20%]

(iii) Sketch the characteristic lines and the trajectories of the cars. [10%]

(b) Consider the initial condition

$$\rho(x,0) = \begin{cases} 0, & x \ge 0 \\ 1, & x < 0 \end{cases}$$

(i) Explain why a similarity solution may exist. [10%]

(ii) Assume a similarity solution is of the form $\rho(x,t) = f(\eta)$, where $\eta = x/t^n$ and n is a constant chosen in order to ensure the similarity solution exists. Find n and f. [20%]

(iii) What is the meaning of this similarity solution? Write down the solution ρ for all x, and $t \ge 0$. [20%]

3 A man wishes to demonstrate a jetpack by jumping off a building with height H, and landing on the ground a specified time T later, with zero velocity. He starts at rest, and, as he descends, his height $h(t)$ above ground is governed by the differential equation

$$\frac{d^2h}{dt^2} = -g + f(t)$$

where $f(t)$ is the upward force generated by the jetpack, and g is the acceleration due to gravity. The cost to run the jetpack is given by:

$$C \propto \int_0^T f(t)^2 \mathrm{d}t$$

(a) In an initial test, the jetpack is off. Find the time to reach the ground and the final velocity. [10%]

(b) Formulate a functional J with minima corresponding to the lowest cost strategy for landing with the jetpack. State clearly what functions J must be minimized with respect to, and what the appropiate boundary conditions are. [20%]

(c) Use variational methods to minimize J, and hence show that the lowest cost strategy requires a jetpack force of the form

$$f = a + bt$$

where a and b are constants. [20%]

(d) Find the complete form of $h(t)$, including finding values for all constants of integration. [30%]

(e) What value of T gives the lowest cost? [20%]

4 A surface of revolution is created by rotating the curve $x = r(z)$ around the z axis. The curve extends between $-a < z < a$, where it has fixed end points, $r(\pm a) = b$, and the surface is closed with disk-shaped caps. It is desired to find the curve $r(z)$ corresponding to the maximum enclosed volume V for a fixed total surface area A.

(a) Show the optimal curve minimizes the functional

$$J = \int_{-a}^{a} \left(r(z)^2 + \lambda r(z) \sqrt{1 + r'(z)^2} \right) \mathrm{d}z$$

and explain the need for the term containing λ. [10%]

(b) Show that the maximizing curve can be written in the form

$$z = \int g(r) \mathrm{d}r$$

and find the form of the function $g(r)$. [40%]

(c) Discuss how the values of the constants in the solution could be found. You do not need to find the values. [10%]

(d) If $r(\pm a)$ is free to vary, find the new boundary conditions for r, and show they are satisfied if $r(\pm a) = 0$. In this calculation, do not neglect the area of the caps. [20%]

(e) The boundaries are fixed at $r(\pm a) = 0$. Find the form of the surface. [20%]

END OF PAPER

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