Crib

4M12 2024, JSB/2

(a) The method of separation of variables can be applied to PDEs when the physical boundaries in space all coincide with constant coordinate lines or surfaces.

(b) Let $f = R(r)X(\theta)$, substitute into the equation,

$$\frac{\partial^2}{\partial r^2}(R(r)X(\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)X(\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)X(\theta)) = 0$$

$$X\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2X}{d\theta^2} = 0$$

separation of variables

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = -\frac{1}{X}\frac{d^2X}{d\theta^2} = k.$$

where k is a constant. Hence

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR = 0, \quad \frac{d^2X}{d\theta^2} + kX = 0$$

k must ≥ 0 because X is a periodic function of θ with period $T = 2\pi$. We set $k = n^2$ with $n \geq 0$. n is integer.

(c) Assume $R = r^\beta$,

$$r^2\beta(\beta-1)r^{\beta-2} + r\beta r^{\beta-1} - n^2 r^\beta = 0$$

$$\Rightarrow \beta^2 - n^2 = 0 \Rightarrow \beta = \pm n$$

Solution r^{-n} is not admissible because it is equal to ∞ at the origin. Hence the solution for X is $X = A\cos(n\theta) + B\sin(n\theta)$

For each $n \geq 0$, there is a pair of R_n and X_n for which $R_n X_n$ satisfy Eqn (1)

Because Eqn (1) is linear, the most general solution is $\sum_{n=0}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$

Q1-2

(d) (i) To satisfy the BC $f(2,\theta) = 2\cos\theta$, $\beta = 1$,

hence $R_1 X_1 = A_1 r\cos(\theta) \Rightarrow A_1 = 1$. The solution is $\boxed{r\cos(\theta)}$

(ii) To satisfy the BC $f(2,\theta) = \sin 2\theta$, $\beta = 2$,

hence $R_2 X_2 = B_2 r^2 \sin(2\theta) \Rightarrow B_2 = \frac{1}{4}$. The solution is

$$\boxed{\frac{1}{4} r^2 \sin(2\theta)}$$

The sum of the above 2 solutions is also a solution, furthermore, it satisfies the BC $f(2,\theta) = 2\cos\theta + \sin 2\theta$

hence the solution is

$$f = r\cos\theta + \frac{1}{4} r^2 \sin(2\theta)$$

(e) g must be periodic with period $T = 2\pi$.

that is, $g(\theta + 2\pi) = g(\theta)$.

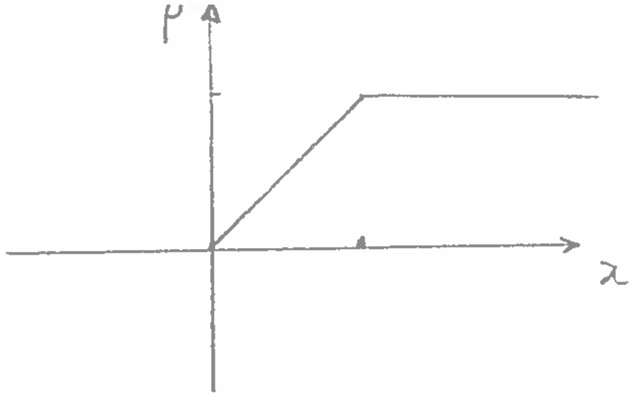
Hence g has a Fourier Series

$$g(\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

For each term in the Fourier series, we can find easily the corresponding pair R & X to satisfy the BC $f(2,\theta) = a_n \cos(n\theta)$ or $f(2,\theta) = b_n \sin(n\theta)$.

The sum of all these solutions X R

is the solution we look for.



(a) Initial condition

$$\frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0$$

(i) characteristic eqn.

$\frac{dx}{dt} = (1-2\rho) = \text{constant} \Rightarrow$

Characteristics are straight lines

There are 3 cases

① IC point $(\xi, 0)$ such as $\xi > 1$, $\rho(\xi, 0) = 1$

characts: $\frac{dx}{dt} = 1 - 2 = -1 \Rightarrow x = \xi - t$, or $x + t = \xi \Rightarrow \rho(\xi,t)=$

② $\xi < 0$, $\rho(\xi, 0) = 0$, characteristics are

$\frac{dx}{dt} = 1 \Rightarrow x = \xi + t$, or $x - t = \xi \Rightarrow \rho(x,t) = 0$

③ for $0 \le \xi \le 1$, $\rho(\xi, 0) = \xi$, characteristics are

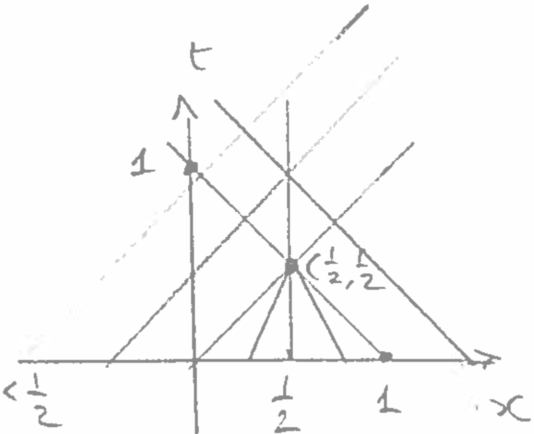
$\frac{dx}{dt} = (1 - 2\xi) \Rightarrow x = (1-2\xi)t + \xi$

However on this characteristic, $\rho(x,t) = \rho(\xi, 0) = \xi$

$$\Rightarrow \rho(x,t) = \frac{x-t}{1-2t}$$

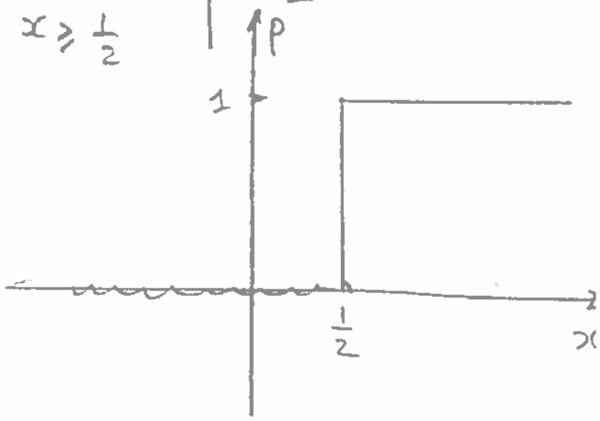
(ii)

Characteristics $\Rightarrow$ converge. Discontinuity forms at $x = \frac{1}{2}$, $t = \frac{1}{2}$



At $t = \frac{1}{2}$ $\rho(x, \frac{1}{2}) = 0$ for $x < \frac{1}{2}$

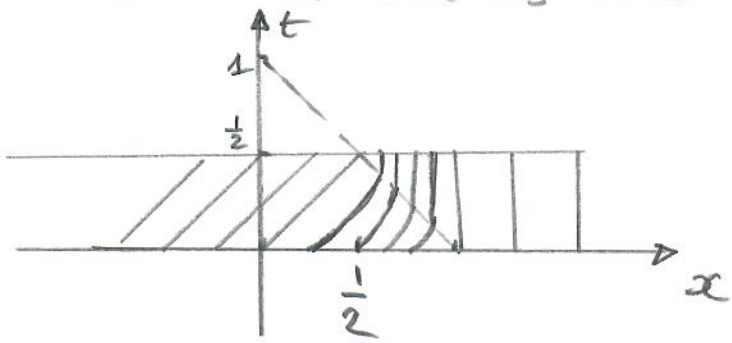
$\rho(x, \frac{1}{2}) = 1$ for $x \geq \frac{1}{2}$



(iii) The characteristics have been shown in (i) : $\frac{dx}{dt} = 1-2\rho$

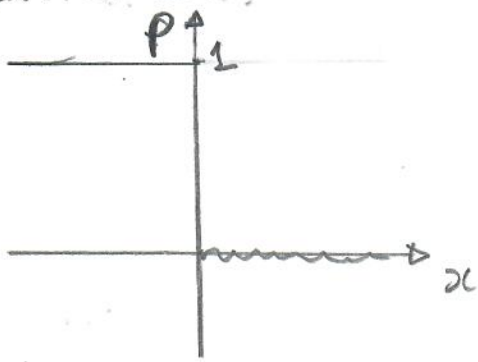
The speed of car is $u = 1-\rho$

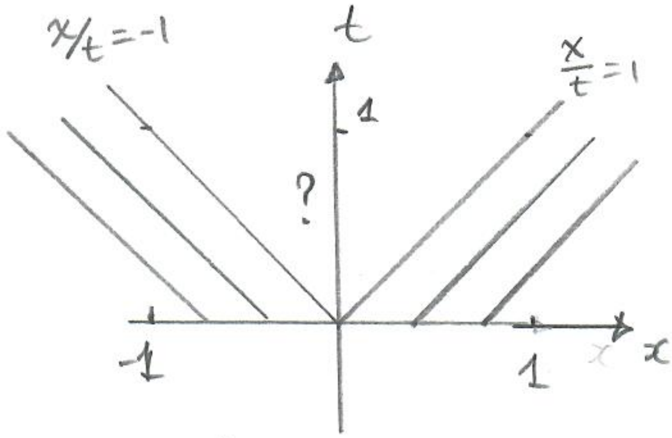
The car trajectories are sketched as below



(b)

(i) Initial condition





$\rho = 1 \quad \frac{dx}{dt} = 1-2\rho = -1$

$\rho = 0 \quad \frac{dx}{dx} = 1-2\rho = 1$

Characteristics diverge

A similarity solution may exist because there is no time & length scale.

$\eta = \frac{x}{t^n}$

(ii) $\frac{\partial \eta}{\partial t} = -\frac{nx}{t^{n+1}} = -\frac{n\eta}{t}$, $\frac{\partial \eta}{\partial x} = \frac{1}{t^n}$,

$P(x,t) = f(\eta)$, substitute into the equation

must chose $n=1$

$$\frac{df}{d\eta}\left(-\frac{n\eta}{t}\right) + (1-2f)\frac{df}{d\eta}\frac{1}{t^n} = 0$$

$$\Rightarrow \boxed{f = \frac{1}{2}(1-\eta)}$$

Q2-3

(iii) the solution is

$$\rho(x,t) = \begin{cases} 0 & \frac{x}{t} \geq 1 \\ \frac{1}{2}\left(1-\frac{x}{t}\right) & -1 \leq \frac{x}{t} \leq 1 \\ 1 & \frac{x}{t} \leq -1 \end{cases}$$

ρ is constant along all rays (straight lines) from the origin.

3 A man wishes to demonstrate a jetpack by jumping off a building with height H, and landing on the ground a specified time T later, with zero velocity. He starts at rest, and, as he descends, his height $h(t)$ above ground is governed by the differential equation

$$\frac{d^2h}{dt^2} = -g + f(t)$$

where $f(t)$ is the upward force generated by the jetpack, and g is the acceleration due to gravity. The cost to run the jetpack is given by:

$$C \propto \int_0^T f(t)^2 \mathrm{d}t$$

(a) In an initial test, the jetpack is off. Find the time to reach the ground and the final velocity. [10%]

Integrating the equation of motion, we have:

$$h = -g\frac{t^2}{2} + c_1 t + c_2.$$

We fix the constants of integration using the initial conditions, $h(0) = H$, $h'(0) = 0$. This gives

$$h = -g\frac{t^2}{2} + H.$$

Solving for $h = 0$, he lands at $T = \sqrt{\frac{2H}{g}}$. His velocity is $h'(T) = -gT = -\sqrt{2Hg}$.

(b) Formulate a functional J with minima corresponding to the lowest cost strategy for landing with the jetpack. State clearly what functions J must be minimized with respect to, and what the appropiate boundary conditions are. [20%]

We have an optimal control problem. We wish to minimize C subject to the constraint that the equation of motion is satisfied, which we implement with a Lagrange multiplier. This brings us to the augmented functional

$$J[f, h, \lambda] = \int_0^T f(t)^2 + \lambda(t)\left(h''(t) + g - f(t)\right) \mathrm{d}t.$$

J must be minimized with respect the three functions, $f(t)$, $\lambda(t)$ and $h(t)$. The only derivative in the integral is $h''(t)$, so h needs boundary conditions for h and h' at $t = 0, T$, while the other functions do not need boundary conditions. The initial conditions on h are determined by the physical state at the start of the jump:

$$h(0) = h \quad h'(0) = 0$$

The end boundary conditions on h are determined by desired state on landing:

$$h(T) = 0 \quad h'(T) = 0.$$

(c) Use variational methods to minimize J, and hence show that the lowest cost strategy requires a jetpack force of the form

$$f = a + bt$$

where a and b are constants. [20%]

Taking the E-L equation for each of the three fields, we get three equations:

$$2f(t) - \lambda(t) = 0 \quad \text{eqn for } f$$

$$\lambda''(t) = 0 \quad \text{eqn for } h$$

$$h''t(t) - g - f(t) = 0 \quad \text{eqn for } \lambda$$

Solving the second equation, we have

$$\lambda = c_3 t + c_4$$

Substituting this into the first equation, we find the form of the force is

$$f = \lambda/2 = c_3 t/2 + c_4/2 = a + bt.$$

as desired.

(d) Find the complete form of $h(t)$, including finding values for all constants of integration. [30%]

Integrating the last E-L equation, we find

$$h(t) = -\frac{1}{2}gt^2 + c_1 t + c_2 + \frac{at^2}{2} + \frac{bt^3}{6}.$$

Applying the initial conditions, we have $c_1 = 0$ *and* $c_2 = H$*, giving:*

$$h(t) = H - \frac{1}{2}gt^2 + \frac{at^2}{2} + \frac{bt^3}{6}.$$

Applying the final conditions requires

$$h(T) = 0 \Rightarrow H/T^2 - \frac{1}{2}g + \frac{a}{2} + \frac{bT}{6} = 0.$$

$$h'(T) = 0 \Rightarrow -g + a + \frac{bT}{2} = 0.$$

Solving these simultaneous equations gives

$$b = 12H/T^3, \qquad a = g - 6H/T^2,$$

which completes the solution.

(e) What value of T gives the lowest cost? [20%]

The motion exactly follows the equation of motion, so we have

$$C \propto J = \int_0^T f(t)^2 \mathrm{d}t.$$

Substituting $f = a + bt$ and integrating, we get

$$J = a^2T + abT^2 + b^2T^3/3.$$

Substituting the results above:

$$\begin{aligned} J &= (g - 6H/T^2)^2T + (g - 6H/T^2)(12H/T^3)T^2 + (12H/T^3)T^3/3 \\ &= g^2T - 12gH/T + 36H^2/T^3 + 12gH/T - 72H^2/T^3 + 48H^2/T^3. \\ &= g^2T + 12H^2/T^3. \end{aligned}$$

Differentiating this with respect to T gives

$$g^2 - 36H^2/T^4.$$

From which we conclude the minimizing time of descent is $T_{min} = \sqrt{6H/g}$.

4 A surface of revolution is created by rotating the curve $x = r(z)$ around the z axis. The curve extends between $-a < z < a$, where it has fixed end points, $r(\pm a) = b$, and the surface is closed with disk-shaped caps. It is desired to find the curve $r(z)$ corresponding to the maximum enclosed volume V for a fixed total surface area A.

(a) Show the optimal curve minimizes the functional

$$J = \int_{-a}^{a} \left(r(z)^2 + \lambda r(z)\sqrt{1 + r'(z)^2} \right) \mathrm{d}z$$

and explain the need for the term containing λ. [10%]

Volume enclosed is given by summing many disk-like cylinders of height dz.*:*

$$V = \int \pi r^2 \mathrm{d}z.$$

Area of the side is given by summing many small frustums of height dz :

$$A = \int 2\pi r\sqrt{1 + r'(z)^2}\mathrm{d}z.$$

We wish to minimize V at fixed A, so we introduce a Lagrange multiplier λ to implement the constraint, leading to the functional

$$\tilde{J} = V + \tilde{\lambda} A$$

Dividing by π, and defining $\tilde{\lambda} = \lambda/2$, we get the functional defined in the question:

$$J = \int_{-a}^{a} F(r, r')\mathrm{d}z = \int_{-a}^{a} r(z)^2 + \lambda r(z)\sqrt{1 + r'(z)^2}\mathrm{d}z.$$

The caps have a fixed area, so do not modify the calculation.

(b) Show that the maximizing curve can be written in the form

$$z = \int g(r)\mathrm{d}r$$

and find the form of the function $g(r)$. [40%]

The functional does not depend explicitly on z, so we can use the Beltrami first integral form of the EL equations:

$$F - r'\frac{\partial F}{\partial r'} = c \qquad (2)$$

Applying this result, we get

$$r^2 + \lambda r\sqrt{1+r'^2} - r'\lambda r\frac{r'}{\sqrt{1+r'^2}} = c \tag{3}$$

$$r\left(r + \frac{\lambda}{\sqrt{1+r'^2}}\right) = c. \tag{4}$$

Where c is a constant of integration. Rearranging for r', we get

$$r'^2 = \frac{\lambda^2}{(c/r - r)^2} - 1 \tag{5}$$

$$\frac{dr}{dz} = \sqrt{\frac{\lambda^2}{(c/r - r)^2} - 1} \tag{6}$$

Dividing by the rhs and integrating w.r.t z, we get the form in the question, with

$$g(r) = \left(\frac{\lambda^2}{(c/r - r)^2} - 1\right)^{-1/2}.$$

(c) Discuss how the values of the constants in the solution could be found. You do not need to find the values. [10%]

The solution can be written as a definite integral

$$z + k = \int_{-a}^{r} g(r)\mathrm{d}r$$

There are thus three constants to find: c, k and λ. These are fixed by the two boundary conditions $r(\pm a) = \pm b$, and the constraint on the total area $A = \int 2\pi r\sqrt{1 + r'(z)^2}\mathrm{d}z$.

(d) If $r(\pm a)$ is free to vary, find the new boundary conditions for r, and show they are satisfied if $r(\pm a) = 0$. In this calculation, do not neglect the area of the caps. [20%]

The area of the caps is $\pi r(a)^2 + \pi r(-a)^2$. Adding this into the functional as part of the area gives us

$$J = \int_{-a}^{a} F(r, r')\mathrm{d}z = \int_{-a}^{a} r(z)^2 + \lambda r(z)\sqrt{1 + r'(z)^2}\mathrm{d}z + \lambda(r(a)^2 + r(-a)^2)/2$$

We form the directional derivative of J,

$$DJ(r, r')[\delta r] = \int_{-a}^{a} 2r\delta r + \lambda\sqrt{1 + r'(z)^2}\delta r + \lambda r\frac{1}{\sqrt{1 + r'(z)^2}}\delta r'\mathrm{d}z + \lambda(r(a)\delta r(a) + r(b)\delta r(b))$$

Following the standard process, we integrate the r' term by parts, producing a total boundary term

$$\lambda r(a)\frac{1}{\sqrt{1 + r'(a)^2}}\delta r(a) + \lambda r(a)\delta r(a) - \lambda r(-a)\frac{1}{\sqrt{1 + r'(-a)^2}}\delta r(-a) + \lambda r(-a)\delta r(-a)$$

We require this to vanish for all $\delta r(a)$ and $\delta r(-a)$ giving the boundary conditions

$$\lambda r(a) + \frac{\lambda r(a)}{\sqrt{1 + r'(a)^2}} = 0$$

and similar for $z = -a$. In both cases this is solved by $r(\pm a) = 0$, although there may be other solutions.

(e) The boundaries are fixed at $r(\pm a) = 0$. Find the form of the surface. [20%]

Returning to the Beltrami form and evaluating at $z = \pm a$, we see that $c = 0$. Therefore, the form of the surface is:

$$z + k = \int \left(\frac{\lambda^2}{r^2} - 1 \right)^{-1/2} \mathrm{d}r = \int_{-a}^{r} r \left(\frac{1}{\lambda^2 - r^2} \right)^{1/2} \mathrm{d}r.$$

We can now conduct the integral to get

$$z + k = -\sqrt{\lambda^2 - r^2}$$

which rearranges into the form

$$(z + k)^2 + r^2 = \lambda^2$$

i.e., the curve is a semi-circle, and the form of the surface is a sphere.

END OF PAPER