Q1-1 (i) when $r \neq 0$

(a) $\nabla\left(\frac{1}{r}\right) = -\frac{e_r}{r^2}$ , $\nabla^2\left(\frac{1}{r}\right) = -\frac{1}{r}\frac{d}{dr}\left(r^2 \times \frac{1}{r^2}\right) = 0$

(ii) $\int_{|x|<a} \nabla^2\left(\frac{1}{r}\right) dV = \int_{|x|=a} \nabla\left(\frac{1}{r}\right) \cdot \vec{n}\, ds = \int_{|x|=a} -\frac{e_r}{r^2} \cdot \underline{e_r}\, ds$

$= -\frac{1}{a^2} \int_{|x|=a} ds = -4\pi$

Hence $\nabla^2\left(-\frac{1}{4\pi|\underline{x}|}\right) = \delta(\underline{x})$

(b) (i) The vector identity

$$\nabla^2 \underline{B} = -\nabla \times (\underbrace{\nabla \times \underline{B}}_{\underline{u}}) + \nabla(\overset{0}{\overset{\shortparallel}{\nabla \cdot \underline{B}}})$$

$$\nabla^2 \underline{B} = -\nabla \times \underline{u} = -\underline{\omega}$$

(ii)

Use the Green function $-\frac{1}{4\pi|\underline{x}|}$

$$\underline{B}(\underline{x}) \; \frac{1}{4\pi} \int \frac{\underline{\omega}(\underline{y})}{|\underline{x}-\underline{y}|} dV(\underline{y})$$

(iii)

$$\underline{u}(\underline{x}) = \nabla_{\underline{x}} \times \underline{B}(\underline{x}) = \frac{1}{4\pi} \int \nabla_x \left(\frac{\underline{\omega}(\underline{y})}{|\underline{x}-\underline{y}|}\right) dV(\underline{y})$$

$$= \frac{1}{4\pi} \int \nabla_x \left(\frac{1}{|\underline{x}-\underline{y}|}\right) \times \underline{\omega}(\underline{y})\, dV(\underline{y})$$

$$= -\frac{1}{4\pi} \int \frac{(\underline{x}-\underline{y}) \times \underline{\omega}(\underline{y})}{|\underline{x}-\underline{y}|^3} dV(\underline{y})$$

Q2-1

(a) The most general semilinear 1st-order PDE is

$$a(x,y)\,u_x + b(x,y)\,u_y = f(x,y,u) \qquad (1)$$

where $f(x,y,u)$ is a continuous function that depends nonlinearly on $u$.

Method of characteristics: consider a special family of curves $C$ which satisfy: $\frac{dx}{ds} = a(x,y)$, $\frac{dy}{ds} = b(x,y)$, (2)

then original PDE (1) can be replaced by the ODE

$$\frac{du}{ds} = f(x,y,u) \qquad (3)$$

Equs (2) are called the characteristic equations of (1)

Eqns (3) the compatibility condition

(b) (i) characteristic $\frac{dy}{dx} = 3 \Rightarrow y = 3x + \xi$

Intersection with $y = \alpha x \Rightarrow \begin{cases} x_0 = \frac{\xi}{\alpha - 3} = \frac{y-3x}{\alpha-3} \\ y_0 = \frac{\alpha\xi}{\alpha-3} \end{cases}$

compatibility condition

$$\frac{du}{dx} = u \Rightarrow \ln u - \ln(x_0) = x - x_0$$

$$\Rightarrow \boxed{u = k(x_0)\,e^{x}}$$

Cauchy condition: $u(x_0, y_0) = \cos(x_0) = k(x_0)\,e^{x_0}$

$x_0 = \frac{y-3x}{\alpha-3} \Rightarrow k(x_0) = \cos\left(\frac{y-3x}{\alpha-3}\right) / e^{\left(\frac{y-3x}{\alpha-3}\right)}$

Hence, $\boxed{u(x,y) = \cos\left(\frac{y-3x}{\alpha-3}\right) e^{\left(\frac{\alpha x - y}{\alpha-3}\right)}}$

(ii) The solution becomes nondifferentiable when $\alpha = 3$, because all charateristics have slop $dy/dx = 3$ so when $\alpha = 3$ the Cauchy data line coincides with a characteristic.

Q1 - 2

(c) (i) characteristic $\frac{dy}{dx} = 1$ , $y = x + \xi$ , $\boxed{\xi = y - x}$

Compatibility $\frac{du}{dx} = u^2$ , $\frac{du}{u^2} = dx$

$$-\frac{1}{u(x,y)} + k(\xi) = x$$

Cauchy condition : $y=0$, $\xi = -x$

$$-\frac{1}{u(-\xi,0)} + k(\xi) = -\xi$$

$$\frac{1}{\tanh(\xi)} + k(\xi) = -\xi$$

$$\boxed{k(\xi) = -\frac{1+\xi\tanh(\xi)}{\tanh(\xi)}}$$

$\xi = y - x$ : $-\frac{1}{u(x,y)} - \frac{1+(y-x)\tanh(y-x)}{\tanh(y-x)} = x$

$$\frac{1}{u(x,y)} = -\frac{1+y\tanh(y-x)}{\tanh(y-x)}$$

$$\boxed{u(x,y) = \frac{\tanh(x-y)}{1-y\tanh(x-y)}}$$

(ii) The solution becomes infinity at $y\tanh(x-y)=1$.

The two asymptotes

① $y=1$ as $x \to +\infty$

② $y \to \infty$, $\tanh(x-y) = \frac{1}{y} \to 0$

$\Rightarrow$ $\tanh(x-y) \sim x - y$

hence $\boxed{x \sim y + \frac{1}{y}}$ $\Rightarrow$ asymptotes $y=x$

3. (a) Lagrangian:

$$L(x,\lambda) = \frac{1}{2}x^T Cx - b^T x + (A\lambda)^T x - \lambda^T f$$

where $\lambda \in \mathbb{R}^m$.

(b) Stationarity:

$$\frac{\partial L}{\partial x} := Cx - b + A\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} := A^T x - f = 0$$

In matrix form,

$$\begin{bmatrix} C & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}$$

The first row $Cx+A\lambda = b$, leading to $x = C^{-1}(b-A\lambda)$. For the second row, $A^T x = f$ Inserting the expressions for $x$, $A^T C^{-1}(b - A\lambda) = f$ which gives $-A^T C^{-1} A\lambda = f - A^T C^{-1} b$. The solvability requirement is that $A$ is full rank so that $A^T C^{-1} A$ is invertible.

(c) If we maximise $L$ over $\lambda$,

$$p(x) := \max_{\lambda} L(x,\lambda) = \begin{cases} \frac{1}{2}x^T Cx - b^T x & \text{if } A^T x = f \\ \infty & \text{otherwise} \end{cases},$$

From $L(x,\lambda) = \frac{1}{2}x^T Cx - b^T x + (A\lambda)^T x - \lambda^T f$,

$$\frac{\partial L}{\partial x} = Cx - b + A\lambda = 0$$

which requires $x = C^{-1}(b - A\lambda)$. Inserting this into $L$, gives $g$:

$$g(\lambda) := -\frac{1}{2}(b - A\lambda)^T C^{-1}(b - A\lambda) - \lambda^T f$$

(d) Minimising $L_{\max_\lambda}$ is equivalent to minimising $x^T Cx - b^T x$ subject to $A^T x = f$. The solution to our problem therefore involves maximising $L$ with respect to $\lambda$ and minimising it with respect to $x$. A *saddle point* is a maximum in one direction and a minimum in the other.

(e) Lagrangian is unchanged,

$$L(x,\lambda) = \frac{1}{2}x^T Cx - b^T x + \lambda^T(Ax - f)$$

but we now require that $\lambda_i \geq 0$. This means that $\lambda$ 'penalises' violations of the inequality constraint.

4.

(a) i. $[\nabla\times u]_k = \epsilon_{klm}\frac{\partial u_m}{\partial x_l}$, therefore $[\nabla\times\nabla\times u]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}\frac{\partial u_m}{\partial x_l}) = \epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}(\frac{\partial u_m}{\partial x_l})$. Applying the $\epsilon_{ijk}$–$\delta_{ij}$ identity, $\epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}(\frac{\partial u_m}{\partial x_l}) = \delta_{il}\delta_{jm}\frac{\partial}{\partial x_j}(\frac{\partial u_m}{\partial x_l}) - \delta_{im}\delta_{jl}\frac{\partial}{\partial x_j}(\frac{\partial u_m}{\partial x_l}) = \frac{\partial}{\partial x_j}(\frac{\partial u_j}{\partial x_i}) - \frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_j}) = \nabla(\nabla\cdot u) - \nabla\cdot(\nabla u)$.

ii. Taking the curl of the $\nabla\times E$ term, $\nabla\times\nabla\times E = \nabla(\nabla\cdot E) - \nabla\cdot(\nabla E) = -\nabla\cdot(\nabla E)$ (since $E$ is divergence-free). Where therefore have $\nabla\cdot(\nabla E) = \frac{\partial}{\partial t}(\nabla\times B) = \mu\epsilon\frac{\partial^2 E}{\partial t^2}$.

We therefore have the vector wave equation $\mu\epsilon\frac{\partial^2 E}{\partial t^2} = \nabla\cdot(\nabla E)$. Following the same process for $B$ yields $\mu\epsilon\frac{\partial^2 B}{\partial t^2} = \nabla\cdot(\nabla B)$

(b) i. $\frac{\partial}{\partial x_i}\epsilon_{ijk}\phi_j u_k = \epsilon_{ijk}\frac{\partial \phi_j}{\partial x_i}u_k + \epsilon_{ijk}\phi_j\frac{\partial u_k}{\partial x_i} = \epsilon_{kij}\frac{\partial \phi_j}{\partial x_i}u_k - \epsilon_{jik}\phi_j\frac{\partial u_k}{\partial x_i} = u\cdot(\nabla\times\phi) - \phi\cdot(\nabla\times u)$.

ii. Introducing $v := \nabla\times u$, we have

$$\int_\Omega w\cdot(\nabla\times v)\ d\Omega + \int_\Omega w\cdot\alpha u\ d\Omega = \int_\Omega w\cdot f\ d\Omega$$

Note that

$$\int_\Omega w\cdot(\nabla\times v)\ d\Omega = -\int_\Omega \nabla\cdot(w\times v)\ d\Omega + \int_\Omega v\cdot(\nabla\times w)\ d\Omega$$

Applying divergence theorem,

$$\begin{aligned}\int_\Omega w\cdot(\nabla\times v)\ d\Omega &= -\int_{\partial\Omega}(w\times v)\cdot n\ d\Gamma + \int_\Omega v\cdot(\nabla\times w)\ d\Omega \\ &= -\int_{\partial\Omega} v\cdot(n\times w)\ d\Gamma + \int_\Omega v\cdot(\nabla\times w)\ d\Omega\end{aligned}$$

The weak form is therefore

$$\int_\Omega(\nabla\times u)\cdot(\nabla\times w)\ d\Omega + \int_\Omega \alpha u\cdot w\ d\Omega = \int_\Omega f\cdot w\ d\Omega$$

iii. From the result in (i), we have

$$\begin{aligned}\int_\Omega(\nabla\times u)\cdot\psi\ d\Omega &= \int_\Omega u\cdot(\nabla\times\psi)\ d\Omega - \int_\Omega\nabla\cdot(\psi\times u)\ d\Omega \\ &= \int_\Omega u\cdot(\nabla\times\psi)\ d\Omega - \sum_i\int_{S_i}(\psi\times u)\cdot n\ d\Gamma \\ &= \int_\Omega u\cdot(\nabla\times\psi)\ d\Omega - \sum_i\int_{S_i}\psi\cdot(n\times u)\ d\Gamma\end{aligned}$$

which requires that the tangential components of $u$ must be continuous across surfaces. *The normal component across surfaces may be discontinuous.*

GNW, v0