Version JL/3

EGT3
ENGINEERING TRIPOS PART IIB

Friday 2 May 2025 14:00 to 15.40

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) Show that $G = -1/(4\pi|\mathbf{x}|)$ is a solution of

$$\nabla^2 G = \delta(\mathbf{x}) \ , \ \ \mathbf{x} \in \mathbf{R}^3 \ ,$$

where $\delta(\mathbf{x})$ is the 3-dimensional delta function: (i) $\delta(\mathbf{x}) = 0$ everywhere except at $\mathbf{x} = 0$, and (ii) $\int_V \delta(\mathbf{x}) dV = 1$ for any volume V which includes $\mathbf{x} = 0$. [30%]

(b) A divergence-free vector field $\mathbf{u}$ is represented by a vector potential $\mathbf{B}$,

$$\mathbf{u} = \nabla \times \mathbf{B} \ .$$

(i) If we make the assumption $\nabla \cdot \mathbf{B} = 0$, show that

$$\nabla^2 \mathbf{B} = -\omega \ ,$$

where $\omega = \nabla \times \mathbf{u}$. [20%]

(ii) Show that

$$\mathbf{B}(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{\omega(\mathbf{y})}{|\mathbf{r}|} \, dV(\mathbf{y}) \ ,$$

where $\mathbf{r} = \mathbf{x} - \mathbf{y}$. In this equation, $\mathbf{x}$ is the position of interest, while $\mathbf{y}$ is the position of the integration element dV and the point where ω from (b)(i) is evaluated. [20%]

(iii) Express $\mathbf{u}$ in terms of an integral involving ω. [30%]

You may use the vector identities in the databook.

2 Consider a scalar function $u(x,y)$, where x and y are two independent variables.

(a) What is a *semilinear* first-order PDE for $u(x,y)$? [10%]

(b) (i) Solve the PDE

$$\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = u$$

for the function $u(x,y)$, subject to the Cauchy condition $u = \cos x$ on the line $y = \alpha x$. α is a constant. [30%]

(ii) Find the value of α for which the method fails and interpret the result. [10%]

(c) (i) Solve the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

for $y > 0$, subject to the Cauchy condition $u(x,0) = \tanh x$. [30%]

(ii) Find the critical curve where the solution becomes infinite. What are the two asymptotes of this curve? [20%]

3 Consider the problem of finding $\mathbf{x} \in \mathbf{R}^n$ that minimises $q(x) = \frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x} - \mathbf{b}^T\mathbf{x}$ subject to the constraint $\mathbf{A}^T\mathbf{x} = \mathbf{f}$, where $\mathbf{C} \in \mathbf{R}^{n\times n}$ is a symmetric positive-definite matrix, $\mathbf{b} \in \mathbf{R}^n$, $\mathbf{A} \in \mathbf{R}^{n\times m}$ is a matrix and $\mathbf{f} \in \mathbf{R}^m$.

(a) Give the Lagrangian $L(\mathbf{x}, \lambda)$ for this problem. [10%]

(b) Give the stationarity conditions and present the matrix system that could be solved to find $\mathbf{x}$, and give the solvability condition for the matrix problem. [40%]

(c) The dual function of $L(\mathbf{x}, \lambda)$ is defined as $g(\lambda) = \min_{\mathbf{x}} L(\mathbf{x}, \lambda)$. We also introduce $p(\mathbf{x}) = \max_{\lambda} L(\mathbf{x}, \lambda)$. Give the expressions for g and p for this problem. [30%]

(d) Problems of this form are sometimes referred to as *saddle point* problems. Explain why. [10%]

(e) If the constraint in the problem was changed to be $\mathbf{A}^T\mathbf{x} - \mathbf{f} \leq 0$, give the Lagrangian and indicate any new restrictions.

[10%]

4 (a) (i) Use index notation to prove that

$$\nabla \times \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}.$$

Note that $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. [20%]

(ii) Consider the slightly simplified version of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Re-formulate the above equations as equivalent wave equations in **E** and **B**. [20%]

(b) (i) Use index notation to show that $\nabla \cdot (\phi \times \mathbf{u}) = \mathbf{u} \cdot (\nabla \times \phi) - \phi \cdot (\nabla \times \mathbf{u})$. [20%]

(ii) Consider the 'curl–curl' equation

$$\nabla \times \nabla \times \mathbf{u} + \alpha \mathbf{u} = \mathbf{f},$$

where α is a constant, on a domain Ω. Pose this problem in its weak form, using $\mathbf{w}$ to denote the weight function and setting $\mathbf{n} \times \mathbf{w} = \mathbf{0}$ on the boundary, where $\mathbf{n}$ is the outward unit normal vector to the boundary. [20%]

(iii) The weak form of the 'curl–curl' problem admits solutions that might not be smooth enough for computing the curl in a classical sense. A weaker requirement is that

$$\int_\Omega (\nabla \times \mathbf{u}) \cdot \psi \, d\Omega = \int_\Omega \mathbf{u} \cdot (\nabla \times \psi) \, d\Omega,$$

where ψ is an infinitely smooth vector-valued function that goes to zero on the boundary of Ω. What continuity conditions on $\mathbf{u}$ across surfaces internal to Ω does this requirement impose? [20%]

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