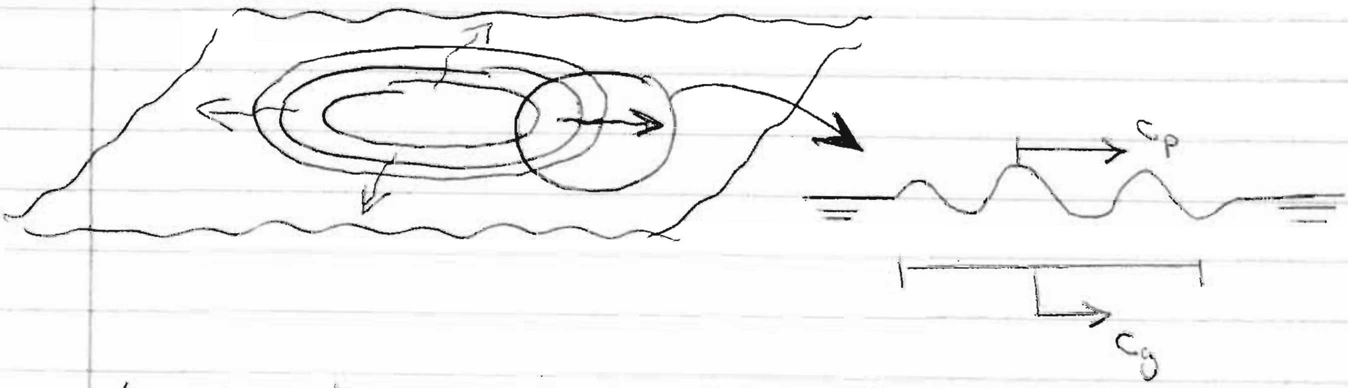
1 (a) $\omega = \sqrt{gk}$

Phase velocity $= \dfrac{\omega}{k} = \sqrt{g/k}$ (c_p)

Group velocity $= \dfrac{\partial \omega}{\partial k} = \frac{1}{2}\sqrt{g/k}$ (c_g)



The overall wave packet travels at c_g which is half the speed of the wave crests, which travel at c_p.

Thus new crests constantly appear at the left of the wave packet, ripple across the wave packet, and then disapear at the right of the wave packet

(b)

$$\frac{\partial^2}{\partial t^2}\nabla^2 \underset{\sim}{u} + (2\underset{\sim}{\Omega}\cdot\nabla)^2 \underset{\sim}{u} = 0$$

Look for plane-wave solutions of form $e^{j(\underset{\sim}{k}\cdot\underset{\sim}{x} - \varpi t)}$

$$(-\varpi^2)(-k^2) + (-2\underset{\sim}{\Omega}\cdot\underset{\sim}{k})^2 = 0 \quad , \quad k = |\underset{\sim}{k}|$$

$$\Rightarrow \quad \varpi^2 = \frac{(2\underset{\sim}{\Omega}\cdot\underset{\sim}{k})^2}{k^2}$$

$$\Rightarrow \quad \varpi = \pm\frac{2\underset{\sim}{\Omega}\cdot\underset{\sim}{k}}{k}$$

$(c_g)_i = \dfrac{\partial \varpi}{\partial k_i}$, Take $\underset{\sim}{\Omega} = \Omega\,\hat{\underset{\sim}{e}}_z$

$$\frac{\partial \omega}{\partial k_x} = \pm 2\,\underset{\sim}{\Omega}\cdot\underset{\sim}{k}\,\frac{\partial}{\partial k_x}\left(\frac{1}{k}\right) = \pm 2(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})\left[\frac{-k_x}{k^3}\right]$$

$$\frac{\partial \omega}{\partial k_y} = \pm 2(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})\left[\frac{-k_y}{k^3}\right]$$

$$\frac{\partial \omega}{\partial k_z} = \pm 2\,\underset{\sim}{\Omega}\cdot\underset{\sim}{k}\left[\frac{-k_z}{k^3}\right] \pm 2\Omega\frac{1}{k}$$

$$\Rightarrow \quad \underset{\sim}{c}_g = \frac{\partial \omega}{\partial k_i} = \pm\frac{2}{k^3}\left[-(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})k_x,\ -(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})k_y,\ -(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})k_z + k^2\Omega\right]$$

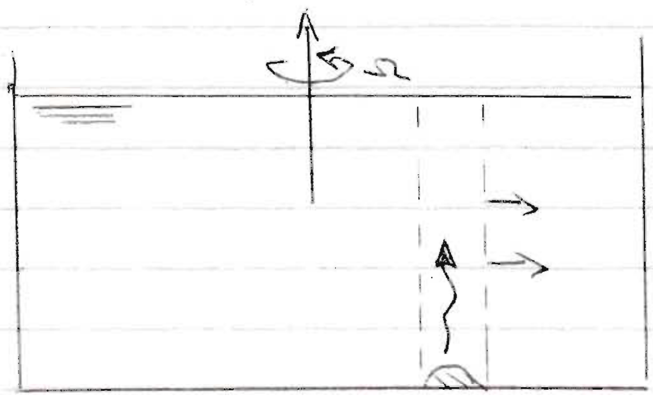
$$= \pm\frac{2}{k^3}\left\{-(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})\underset{\sim}{k} + k^2\underset{\sim}{\Omega}\right\}$$

$$= \pm\frac{2}{k^3}\left\{\underset{\sim}{k}\times(\underset{\sim}{\Omega}\times\underset{\sim}{k})\right\}$$

$$\bar{\omega} = \pm 2(\underset{\sim}{\Omega}\cdot\underset{\sim}{k})/k \Rightarrow \text{low-frequency waves have } \underset{\sim}{\Omega}\cdot\underset{\sim}{k}\approx 0$$

$$\Rightarrow \quad \underset{\sim}{c}_g \approx \pm\frac{2\underset{\sim}{\Omega}}{k}$$

(c)



Slowly moving object emits low-frequency waves. These travel upward with a group velocity of $|\underset{\sim}{c}_g| = 2\Omega L$ where L is the size of the object. The waves carry the information that the object is moving.

2 (a) The solution is self-similar when there is no geometric length scale in the problem, i.e. solution takes the form

$$f\left(\frac{x}{l(t)}\right), \quad l = \text{diffusion length}.$$

(b)

Long rod $\Rightarrow$ $\underset{m}{l} = f(\underset{m^2/s}{\nu}, \underset{s}{t})$ [no other variables]

By inspection $l \sim \sqrt{\nu t}$

$$\frac{T - T_0}{\Delta T} = F(\underset{m}{x}, \underset{m^2/s}{\nu}, \underset{s}{t}) \quad \text{[No other variables]}$$

Π Theorem: 3 variables, 2 dimensions $\Rightarrow$ 1 dimensionless group.

By inspection, dimensionless group is

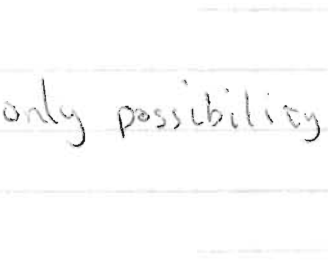
$$\frac{x}{\sqrt{\nu t}} = \frac{x}{l(t)}$$

Since $(T - T_0)/\Delta T$ is dimensionless, only possibility is

$$\underline{\frac{T - T_0}{\Delta T} = F\left(\frac{x}{\sqrt{\alpha t}}\right)}$$

(c) In spherical polars

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + 2\frac{1}{r}\frac{\partial T}{\partial r}\right]$$



Let $y = r - R$, $\Pi = r[T - T_0]$

$$\frac{\partial^2 \Gamma}{\partial y^2} = \frac{\partial^2}{\partial r^2}\left(r(T-T_0)\right) = \frac{\partial^2}{\partial r^2}(rT) = \frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} + T\right)$$

$$= r\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right)$$

$$= r\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r}\right]$$

Thus,
$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r}$$

becomes
$$\frac{\partial}{\partial t}(rT) = \alpha \frac{\partial^2 \Gamma}{\partial y^2}$$

or
$$\underline{\underline{\frac{\partial \Gamma}{\partial t} = \alpha \frac{\partial^2 \Gamma}{\partial y^2}}}$$

In terms of Γ and y there is no imposed geometric scale, so solution must be of the form

$$\underline{\underline{\frac{\Gamma}{R\Delta T} = F\left(\frac{y}{\sqrt{\alpha t}}\right)}} \Rightarrow \frac{\partial F}{\partial t} = \alpha \frac{\partial^2 F}{\partial y^2}$$

dimensionless

$$\frac{dF}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}F'(\eta)$$

Thus,
$$-\frac{1}{2}\frac{\eta}{t}F'(\eta) = \alpha\frac{\partial^2 F}{\partial y^2} = \alpha F''(\eta)\frac{1}{\alpha t}$$

$$\Rightarrow \underline{\underline{F''(\eta) + \frac{1}{2}\eta F'(\eta) = 0}}$$

(d) Integrate once: $F'(\eta) = B e^{-\eta^2/4} \Rightarrow F = A + B\int_0^\eta e^{-\eta^2/4}\, d\eta$

In terms of T: $\dfrac{\Gamma}{R\Delta T} = \dfrac{r(T-T_0)}{R\,\Delta T} = A + B\int_0^\eta e^{-\eta^2/4}\, d\eta$

B.C.: at $r = R$, $T - T_0 = \Delta T = A = 1$
at $r \to \infty$, $T - T_0 = 0 \Rightarrow A + B\underbrace{\int_0^\infty e^{-\eta^2/4}\, d\eta}_{\sqrt{\pi}} = 0 \Rightarrow B = -\frac{1}{\sqrt{\pi}}$

$$\Rightarrow \frac{T-T_0}{\Delta T} = \frac{R}{r}\left[1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2/4}\, d\eta\right]$$

Question 3

(a) Multiply the Eqn by the variation of the dependent variable, $\delta u(x)$, and integrate over $[0, 1]$

$$\int_0^1 (u'' - u + x)\, \delta u \, dx = 0$$

Integration by part & apply the B.C.s

$$\int_0^1 (-u' \delta u' - u \delta u + x \delta u)\, dx = 0$$

$$\delta \int_0^1 (u')^2 + u^2 - 2xu \, dx \qquad (3.1)$$

Variational functional $\quad I(u) = \int_0^1 (u')^2 + u^2 - 2xu \, dx$.

(b) $\quad \bar{u} = C_1 x(1-x)$, $\quad \bar{u}' = C_1(1-2x)$

Substituting into (3.1) and integrating leads to

$$\delta \int_0^1 C_1^2 (1-2x)^2 + C_1^2 x^2 (1-x)^2 - 2C_1 x^2(1-x)\,]dx = 0$$

$$\delta \left[\frac{11}{30} C_1^2 - \frac{1}{6} C_1 \right] = 0$$

$$\frac{11 C_1}{15} \delta C_1 - \frac{1}{6} \cancel{\delta C_1} = 0 \implies C_1 = \frac{5}{22}$$

(c) Let ϕ be a trial function over $[0, 1]$, multiply the Eqn by ϕ, and integrate by part

$$\int_0^1 (u'' - u + x)\, \phi \, dx = 0$$

$$\underline{\int_0^1 -(u'\phi' + u\phi) + x\phi \; dx}$$

$$\phi' = 1 - 2x$$

(d) For $i = 1$, with trial function $\phi_1(x) = x(1-x)$, we have

$$\int_0^1 \{ -[(1-2x)C_1 + (2x - 3x^2) C_2][1 - 2x] - [(x - x^2)C_1 + (x^2 - x^3)C_2](x - x^2)$$
$$- x(x^1 - x^2) \} \, dx = 0$$

Multiplying out the polynomials & collecting terms yields

$$\int_0^1 [(-1+4x-5x^2+2x^3-x^4)c_1 + (-2x+7x^2-7x^3+2x^4-x^5)c_2$$

$$+ x^2-x^3]\, dx = 0$$

$$\Rightarrow \quad 22c_1 + 11c_2 = 5 \qquad (3.2)$$

For $i=2$, $\phi_2 = x^2-x^3$, $\phi_2' = 2x-3x^2$

with ϕ_2 as trial function, we have

$$\int_0^1 \{-[(1-2x)c_1 + (2x-3x^2)c_2](2x-3x^2) - [(x-x^2)c_1 + (x^2-x^3)c_2]$$

$$(x^2-x^3) + x(x^2-x^3)\, dx$$

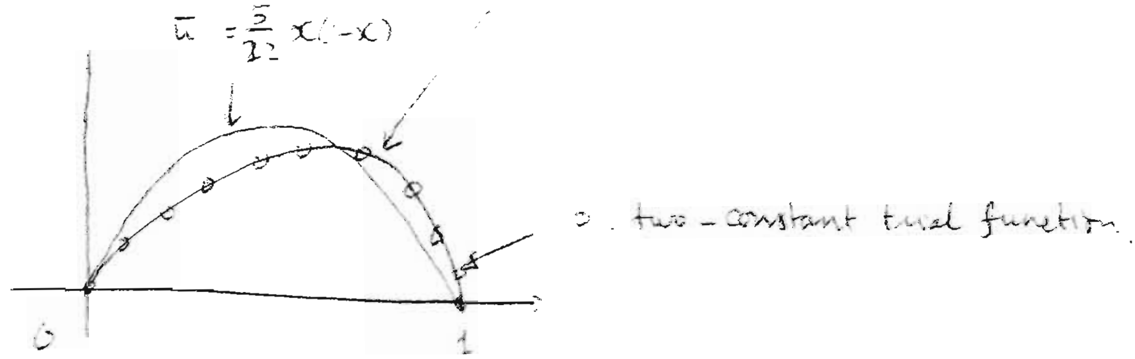
$$= \int_0^1 - c_1(2x-7x^2+7x^3-2x^4+x^5) - c_2\,(4x^2-12x^3+10x^4$$

$$-2x^5+x^6) + (x^3-x^4)\, dx$$

$$\Rightarrow \quad \frac{11}{60}c_1 + \frac{1}{7}c_2 = \frac{1}{20}$$

Solving (3.2) and (3.3), we obtain $c_1 = \frac{69}{473}$, $c_2 = \frac{7}{43}$

(e) The exact solution is $u = \frac{e}{e^2-1}(e^{-x}-e^{x})+x$



The two-constant trial function agrees well with the exact solution

Question 4

$$DI(u)[v] = \lim_{\varepsilon\to 0}\left[\frac{1}{2}\left(u(1)+\varepsilon v(1)\right)^2 + \frac{1}{2}\int_0^1 (u'+\varepsilon v')^2 dx\right.$$
$$= u(1)v(1) + \int_0^1 u'v'dx$$

(a) $\delta I = u(1)\,\delta u(1) + \int_0^1 u'\,\delta u'\,dx = 0$

where we note that $u(1)$ does vary in this context.

(b) Integrating by part,

$$u'(1)\delta u(1) - \underbrace{\cancel{u'(0)\delta u(0)}}_{=0} + u(1)\,\delta u(1) - \int_0^1 u''\delta u\,dx = 0.$$

From the integrant, the Euler Eqn is $\boxed{u''=0}$ (4.1)

(c) Boundaray condition at $x=0$ $\boxed{u(0)=1}$ (4.2)

boundary condition at $x=1$, the coefficient of the variation δu must vanish.

$$\boxed{u'(1)+u(1)=0} \quad (4.3)$$

(d) The solution ~~to~~ (4.1) is a straight line:

$$u(x) = c_1 x + c_2$$

$u(0)=1 \Rightarrow c_2 = 1$, and

$u(1)+u'(1)=0 \Rightarrow c_1 + c_2 + c_1 = 0 \Rightarrow c_1 = -\frac{1}{2}$

Hence the stationary function is

$$u(x) = -\frac{1}{2}x + 1$$