Crib

4M12 2021, JSB/2

1 (a) Non-dispersive waves have a phase speed which is <u>indepedent</u> of wavenumber, whereas dispersive waves have a phase speed that is a function of wavenumber. For non-dispersive waves, all Fourier modes travel at the same speed. This means that, if an initial disturbance is decomposed into many Fourier modes, each mode travels at the same speed and so all modes travel the same distance in a given time. Hence disturbances travel without change of shape in a non-dispersive system (ignoring friction). On the other hand, in a dispersive system, disturbances change shape as they propagate as different Fourier modes travel at different speeds.

(b)
$$\phi(x,t) = \int a(k) \exp[i(kx - \varpi t)]\, dk$$

$$\varpi(k) = \varpi(k_0) + (k-k_0)\left(\frac{d\varpi}{dk}\right)_0 + O(k-k_0)^2$$

Since $a(k) \approx 0$ for $k \neq k_0$ may truncate the Taylor expansion at linear order.

$$\Rightarrow \phi(x,t) = \int a(k) \exp\left[i\left((k-k_0)x + k_0 x - \varpi_0 t - \frac{d\varpi}{dk}(k-k_0)t\right)\right]$$

$$= e^{i(k_0 x - \varpi_0 t)} \int a(k) \exp\left[i\left(x - \left(\frac{d\varpi}{dk}\right)_0 t\right)(k-k_0)\right] dk$$

(since k_0 and ϖ_0 constants)

$$= e^{i(k_0 x - \varpi_0 t)} \int a(k^*) \exp\left[i\left(x - \left(\frac{d\varpi}{dk}\right)_0 t\right) k^*\right] dk^*$$

$(k^* = k - k_0)$

$$= \exp[i(k_0 x - \varpi_0 t)]\, A\left(x - \left(\frac{d\varpi}{dk}\right)_0 t\right)$$

(since k^* is a dummy variable)

(b) cont.

$$\phi(x,t) = \underbrace{A\left(x - \left(\frac{d\varpi}{dk}\right)_0 t\right)}\exp\left[i(k_0 x - \varpi_0 t)\right]$$

Amplitude function travels at speed $(d\varpi/dk)_0$

$$\Rightarrow \quad \underline{\underline{\text{group velocity} = \frac{d\varpi}{dk}}}.$$

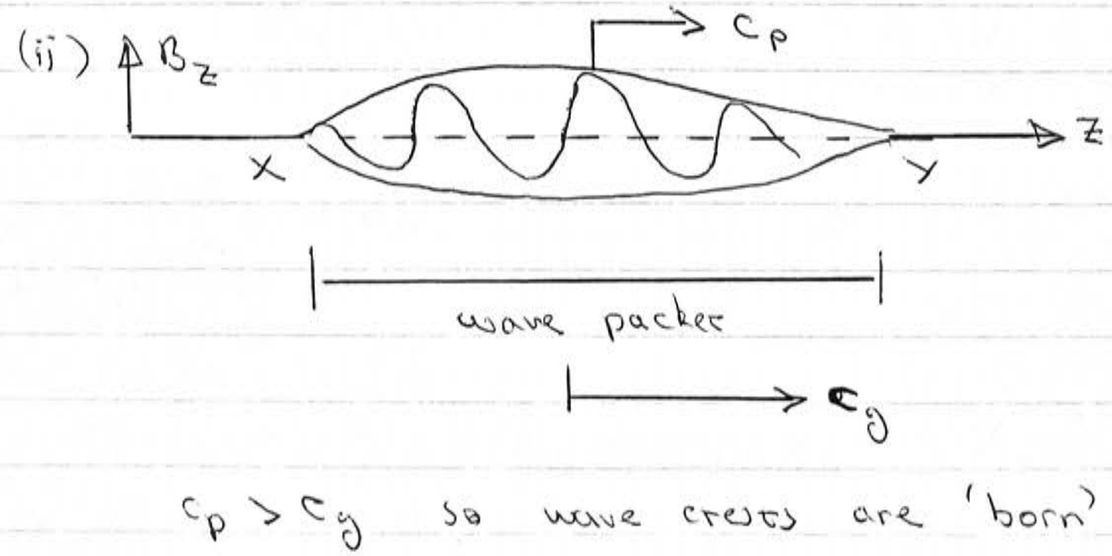
(c) (i) $\omega^2 = \varpi_{min}^2 + c^2k^2 \Rightarrow 2\varpi \frac{d\varpi}{dk} = 2c^2k$

$$\Rightarrow c_g = \frac{d\varpi}{dk} = \frac{c^2k}{\varpi} = \underline{\underline{\frac{c\sqrt{\varpi^2 - \varpi_{min}^2}}{\varpi}}}$$

$$c_p = \frac{\varpi}{k} = \frac{c\varpi}{ck} = \underline{\underline{\frac{c\,\varpi}{\sqrt{\varpi^2 - \varpi_{min}^2}}}} \quad (>c!)$$

$$\Rightarrow \underline{\underline{c_p c_g = c^2}}$$

(ii)



$c_p > c_g$ so wave crests are 'born' at X and die at Y

2 (a)

$$\nabla \cdot \underset{\sim}{E} = \rho/\epsilon_0 , \quad \underset{\sim}{E} = -\nabla V \Rightarrow \nabla^2 V = -\rho/\epsilon_0$$

(i)

For sphere centred on q, radius r,

$$\int \nabla \cdot \underset{\sim}{E}\, dv = \oint \underset{\sim}{E} \cdot d\underset{\sim}{s} = E_r 4\pi r^2 = \frac{1}{\epsilon_0} \underbrace{\int \rho\, dv}_{q}$$

$$\Rightarrow E_r = \frac{q}{4\pi\epsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{q}{4\pi\epsilon_0 r^2} \Rightarrow V = \frac{q}{4\pi\epsilon_0 r}$$

(ii) For charge at $\underset{\sim}{x}'$

$$V = \frac{q}{4\pi\epsilon_0 |\underset{\sim}{x} - \underset{\sim}{x}'|} \quad \text{(shift origin)}$$

Apply superposition:

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}'$$

(Green's function solution of $\nabla^2 V = -\rho/\epsilon_0$)

(iii) $\underset{\sim}{E} = -\nabla V$

But $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{e}_r}{r^2} = -\frac{\underset{\sim}{r}}{r^3}$

$$\Rightarrow \nabla\left(\frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\right) = \frac{-(\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} \quad \text{(change of origin)}$$

$$\Rightarrow \quad E(\underset{\sim}{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\underset{\sim}{x}')(\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} d\underset{\sim}{x}'$$

(iv) Each component of $\underset{\sim}{A}$ ~~two~~ inverts like V in part (a). Hence

$$A_i = \frac{1}{4\pi} \int \frac{s_i(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}'$$

(b) $\nabla^2 \underset{\sim}{A} = -\mu_0 \underline{J}(x)$

From part (b), $\underset{\sim}{A}(\underset{\sim}{x}) = \frac{\mu_0}{4\pi} \int \frac{\underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \, d\underset{\sim}{x}'$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left[\frac{\underset{\sim}{J}'}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \right] d\underline{x}'$$

$$= \frac{\mu_0}{4\pi} \int \nabla \left(\frac{1}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \right) \times \underset{\sim}{J}(\underline{x}') \, d\underset{\sim}{x}'$$

(since $\underset{\sim}{J}(\underline{x}')$ is a constant w.r.t. $\underset{\sim}{x}$)

But $\nabla \left(\frac{1}{|\underset{\sim}{x}-\underline{x}'|} \right) = \frac{-(\underset{\sim}{x}-\underline{x}')}{|\underset{\sim}{x}-\underline{x}'|^3}$ (from part (a))

$$\Rightarrow \quad \underset{\sim}{B} = -\frac{\mu_0}{4\pi} \int \frac{(\underset{\sim}{x}-\underline{x}')}{|\underset{\sim}{x}-\underline{x}'|^3} \times \underset{\sim}{J}' \, d\underline{x}' = \frac{\mu_0}{4\pi} \int \frac{\underset{\sim}{J}' \times \underset{\sim}{r}}{|\underset{\sim}{r}|^3} \, d\underset{\sim}{x}'$$

$(\underset{\sim}{r} = \underset{\sim}{x} - \underline{x}')$

$$\nabla \cdot \underset{\sim}{B} = -\frac{\mu_0}{4\pi} \nabla \cdot \int \underset{\sim}{J}' \times \nabla \left(\tfrac{1}{r}\right) dV' = \frac{\mu_0}{4\pi} \int \underset{\sim}{J}' \cdot \nabla \times \left(\nabla \left(\tfrac{1}{r}\right)\right) d\underset{\sim}{x}' = 0$$

(c) Retarded potential soln of

$$\nabla^2 \underset{\sim}{A} - \frac{1}{c^2} \frac{\partial A}{\partial t} = -\mu_0 \underset{\sim}{J}(\underset{\sim}{x}, t)$$

is $\underset{\sim}{A}(\underset{\sim}{x}, t) = \frac{\mu_0}{4\pi} \int \frac{\underset{\sim}{J}(\underset{\sim}{x}', t - |\underset{\sim}{r}|/c)}{|\underset{\sim}{r}|} \, d\underset{\sim}{x}'$

$$\underset{\sim}{r} = \underset{\sim}{x} - \underline{x}'$$

The time lag of $|\underset{\sim}{r}|/c$ is because it takes a time $|\underset{\sim}{r}|/c$ for information about the value of $\underset{\sim}{J}$ at $\underline{x}'$ to travel to point $\underset{\sim}{x}$.

Pressure waves in acoustics governed by same inhomogeneous wave equation:

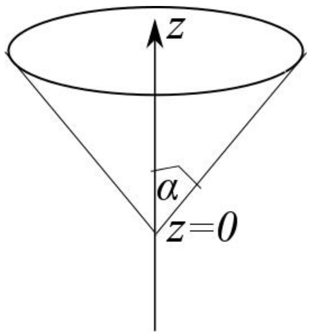
$$\nabla^2 \rho - \frac{1}{a^2} \frac{\partial^2 \rho}{\partial t^2} = \text{source}$$

where ρ = density and a = speed of sound.

3 *A surface is given by $z(r,\theta) = r\cot\alpha$, where $r-\theta-z$ are cylindrical coordinates and $0 < \alpha < \pi/2$ is a constant.*

(a) *Sketch the surface, and give a geometrical interpretation of α.* *[10%]*

There is no θ dependence, so this is a surface of revolution. We have $z \propto r$ so the surface is a straight line in the $r-z$ plane, i.e. the surface is a cone. The constant α is the half angle of the cone.



(b) *Find an expression for the length of the path along the surface described by the function $r = f(\theta)$, starting from θ_1 and finishing at θ_2.* *[20%]*

In cylindrical coordinates, the infinitesimal path length (by Pythagorus) is

$$\mathrm{d}l = \sqrt{\mathrm{d}r^2 + \mathrm{d}z^2 + r^2\mathrm{d}\theta^2}.$$

Along our path, we have $r = f(\theta)$, $z = f(\theta)\cot\alpha$, giving

$$\mathrm{d}l = \sqrt{\mathrm{d}f^2 + \mathrm{d}f^2\cot^2\alpha + f^2\mathrm{d}\theta^2}.$$

Integrating along the path, and pulling $d\theta$ out of the square root, we get the total length

$$l = \int_{\theta_1}^{\theta_2} \sqrt{(1+\cot^2\alpha)f'^2 + f^2}\mathrm{d}\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{f'^2\csc^2\alpha + f^2}\mathrm{d}\theta.$$

(c) *Using the Beltrami identity, or otherwise, show that f will extremise the path length if the quantity* [30%]

$$\frac{f^2}{\sqrt{f^2 + \csc^2(\alpha)f'^2}} = 0$$

is constant along the path.

We wish to exremise l over variations in $f(\theta)$. We note that the integrand $F = \sqrt{f'^2 \csc^2\alpha + f^2}$ does not depend on explicitly on θ so, rather than use the usual Euler-Lagrange equation, we use the Beltrami modified form.

$$-\frac{\partial F}{\partial \theta} + \frac{\mathrm{d}}{\mathrm{d}\theta}\left(F - f'\frac{\partial F}{\partial f'}\right) = 0.$$

The first term vanishes, and the term in brackets evaluates to

$$F - f'\frac{\partial F}{\partial f'} = \sqrt{f'^2 \csc^2\alpha + f^2} - f'\frac{f'\csc^2\alpha}{\sqrt{f'^2\csc^2\alpha + f^2}} = \frac{f^2}{\sqrt{f'^2\csc^2\alpha. + f^2}}$$

The condition for extreme path length is thus simply

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{f^2}{\sqrt{f'^2\csc^2\alpha + f^2}}\right) = 0.$$

which does indeed imply that the quantity in brackets is constant along the path.

(d) *Find and sketch the shortest path along the surface from $(r,\theta) = (r_0, -\beta)$ to (r_0, β), and show that the minimum value of r along this path is* *[40%]*

$$r_{min} = r_0 \cos(\beta \sin(\alpha)).$$

Integrating our equation once gives:

$$\frac{f^2}{\sqrt{f'^2\csc^2\alpha + f^2}} = c_1.$$

Rearranging this equation for f' gives

$$f' = \sin\alpha\sqrt{f^4/c_1^2 - f^2}.$$

We can now divide by $\sqrt{f^4 - c_1^2 f^2}$ and integrate a second time to get

$$\int \frac{1}{\sqrt{f^4/c_1^2 - f^2}}\mathrm{d}f = (\theta - \theta_0)\sin\alpha.$$

Using the hint, the integral on the left gives

$$\sec^{-1}(f/c_1) = (\theta - \theta_0)\sin\alpha,$$

which we can solve for f to get

$$f = c_1 \sec((\theta_0 - \theta)\sin\alpha).$$

We must use the end points to fix c_1 and θ_0. From symmetry, we expect r_{min} at $\theta = 0$. This occurs when the argument of sec is 0, so we set $\theta_0 = 0$ giving

$$f = c_1 \sec(\theta \sin\alpha).$$

Finally, we need $f = r_0$ at $\theta = \pm\beta$, requiring $c_1 = r_0 \cos(\beta \sin\alpha)$. The full path is thus

$$f = r_0 \cos(\beta \sin\alpha) \sec(\theta \sin\alpha)$$

and the closest approach is at $\theta = 0$, where $f = r_0 \cos(\beta \sin\alpha)$.

4 (a) *Consider the following equation for $u(x)$, which is to be solved for $0 < x < 1$,*

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \frac{2}{x}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = 0, \qquad \begin{array}{l} u(0) = 0,\ u'(0) = 0 \\ u(1) = 0,\ u'(1) = 0. \end{array}$$

(i) *Find a weak form of the equation, and explain why it is not possible to deduce a variational form.* [20%]

We multiply by an arbitrary weight function $w(x)$, with $w(0) = w(1) = w'(0) = w'(1) = 0$, and integrate over the domain.

$$\int_0^1 wu'''' + 2wx^{-1}u'''\mathrm{d}x = 0$$

Integrating the first term by parts twice and the second term by parts once gives the form with the lowest order derivatives:

$$\int_0^1 w''u'' - 2w'x^{-1}u'' + 2wx^{-2}u''\mathrm{d}x = 0$$

All boundary terms vanish due to the boundary conditions on w. This is not symmetric in w and u, so we cannot deduce a variational form.

(ii) *Multiply the original equation by x, find the new weak form, and hence find an equivalent variational form.* [20%]

The new equation is

$$xu'''' + 2u''' = 0.$$

We again multiply by an arbitrary weight function and integrate over the domain.

$$\int_0^1 wxu'''' + 2wu'''\mathrm{d}x = 0$$

Integrating the first term by parts twice and the second onece gives

$$\int_0^1 -w'xu''' - wu''' - 2w'u''\mathrm{d}x = 0$$

$$\int_0^1 w''xu'' + w'u'' + w'u'' - 2w'u''\mathrm{d}x = 0$$

$$\int_0^1 w''xu''\mathrm{d}x = 0$$

This is symmetric, and is the directional derivative $Dxu''u''(u)[w]$. Solving the original equation is thus equivalent to extremising $\int_0^1 x(u'')^2\mathrm{d}x$.

(b) *Anna has recently completed her engineering degree, and is making financial plans. She starts her career with no savings, $S(0) = 0$, but in forty years time she will need $S(40) = S_F$ to retire comfortably. Fortunately Anna secures a good job, which provides a constant income I. She also invests her savings, S, in an account paying an interest rate r, (generating an additional income rS) and consumes (spends) at a rate $C(t)$, such that, overall, her savings grow as*

$$\dot{S} = rS + I - C.$$

(i) *Show that Anna's savings target is equivalent to the integral constraint* [20%]

$$\int_0^{40} \mathrm{e}^{r(40-t)}(I - C(t))\mathrm{d}t = S_F.$$

We have the differential equation

$$\dot{S} - rS = I - C(t).$$

Using an integrating factor, with $P(t) = -r$ and $Q(t) = I - C$, and $\int P(t)\mathrm{d}t = -rt$.

$$\frac{d}{dt}\left(S\mathrm{e}^{-rt}\right) = (I - C(t))\mathrm{e}^{-rt}.$$

Integrating both sides with respect to t, we have

$$S(t)\mathrm{e}^{-rt} = \int_0^t (I - C(t))\mathrm{e}^{-rt}\mathrm{d}t + c_1.$$

The initial condition $S(0) = 0$ sets $c_1 = 0$, while $S(40) = S_F$ then gives

$$S_F\mathrm{e}^{-40r} = \int_0^{40} (I - C(t))\mathrm{e}^{-rt}\mathrm{d}t \quad \to \quad S_F = \int_0^{40} (I - C(t))\mathrm{e}^{r(40-t)}\mathrm{d}t.$$

(ii) *Anna wishes to pace her consumption (spending), $C(t)$ over her career in order to maximize her utility while still hitting her retirement savings target. To do this, she decides to maximize the quantity*

$$U = \int_0^{40} \log\left(C/C_0\right)\mathrm{d}t$$

where C_0 is a (constant) level of spending required for basic subsistence. where C_0 is a (constant) level of spending required for basic subsistence. Show that, to maximize U while meeting her retirement savings target, Anna should choose $C(t)$ of the form

$$C(t) = \frac{1}{\lambda}\mathrm{e}^{r(t-40)},$$

and find an expression for the constant λ *[40%]*

Introducing the Lagrange multiplier λ leads us to the modified functional

$$\tilde{U} = \int_0^{40} \log(C/C_0) + \lambda \left(\mathrm{e}^{r(40-t)}(I-C) - S_F/40 \right) \mathrm{d}t.$$

Taking the directional derivative with respect to C, we get

$$D\tilde{U}(C)[v] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^{40} \log((C+\varepsilon v)/C_0) + \lambda \left(\mathrm{e}^{r(40-t)}(I-(C+\varepsilon v)) - S_F/40 \right) \mathrm{d}t \bigg|_{\varepsilon=0} = 0.$$

Which evaluates to:

$$\int_0^{40} \frac{v}{C} + \lambda \left(\mathrm{e}^{r(40-t)} v \right) \mathrm{d}t = 0.$$

Since this quantity must vanish for all v, we get the condition

$$\frac{1}{C} - \lambda \left(\mathrm{e}^{r(40-t)} \right) = 0$$

which we solve for C to get

$$C = \frac{1}{\lambda} \mathrm{e}^{r(t-40)}.$$

To fix the value of λ, we substitute this form into the constraint,

$$\int_0^{40} \mathrm{e}^{r(40-t)} \left(I - \frac{1}{\lambda} \mathrm{e}^{r(t-40)} \right) \mathrm{d}t = S_F.$$

$$\int_0^{40} \left(I \mathrm{e}^{r(40-t)} - \frac{1}{\lambda} \right) \mathrm{d}t = S_F.$$

$$\frac{I}{r} \left(\mathrm{e}^{40r} - 1 \right) - \frac{40}{\lambda} = S_F \qquad \rightarrow \qquad \frac{1}{\lambda} = \frac{I}{40r} \left(\mathrm{e}^{40r} - 1 \right) - \frac{S_F}{40}$$

so the final answer is

$$C(t) = \left(\frac{I}{40r} \left(\mathrm{e}^{40r} - 1 \right) - \frac{S_F}{40} \right) \mathrm{e}^{r(t-40)}.$$

Examiners comments. Statistics refer to the IIB cohort.

Q1. 37 attempts, mean 13.11, std 4.32. This question is on Green's function solutions and their relationship to the Biot-Savart law and to retarded potentials. It was attempted by all candidates and the results were a little disappointing. Some candidates struggled with the basic vector calculus required for the question.

Q2. 33 attempts, mean 14.91, std 2.89. This question was on dispersive waves, wave packets and the concept of group velocity. It was attempted by all but 4 candidates and the results were gratifying. Most candidates displayed a clear grasp of the fundamental ideas of group velocity and wave packets.

Q3. 18 attempts, mean 15.50, std 4.74. Least popular question, but done well. Almost all candidates identified the conical surface. Most candidates could also find an expression for the path length though many failed to include dz^2. Part (c) was uniformly well answered. However, many marks were dropped in (d), either during solving the differential equation, or applying the boundary conditions. Surprisingly few candidates finished with a diagram of a geodesic on a cone.

Q4. 24 attempts, mean 12.38 std 3.93. Fairly popular question and mostly done well. Candidates scored well in (a) (week form) though marks were lost for failing to discuss boundary conditions, and for answers containing third derivatives. (bi) was quite poorly answered, despite being amenable to a simple integrating factor or a Laplace transform. (bii) was well answered, with almost all candidates introducing a Lagrange multiplier, and little confusion being caused by the integral depending on C(t) but not its derivatives. However, fairly few candidates persisted to the end of (bii), and found the value of the Lagrange multiplier.