Version JSB/5

EGT3/EGT2
ENGINEERING TRIPOS PART IIB
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Friday 30 April 2021 1.30 to 3.10

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) An electrostatic field $\mathbf{E}$ and its scalar potential V are governed by

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \mathbf{E} = -\nabla V,$$

where ϵ_0 is the permittivity of free space and $\rho(\mathbf{x})$ is the charge density.

(i) For a charge q sitting at the origin, $\mathbf{E}$ has only a radial component, E_r. Find expressions for $E_r(r)$ and $V(r)$ where r is the distance from the origin. [10%]

(ii) Use superposition to find a general expression for $V(\mathbf{x})$ valid for any localised charge distribution $\rho(\mathbf{x})$. [10%]

(iii) Show that

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3},$$

where ∇ operates on $\mathbf{x}$ but not $\mathbf{x}'$, and hence find an expression for $\mathbf{E}(\mathbf{x})$ valid for any localised charge distribution. [10%]

(iv) Use the results of part (ii) above to show that the Green's function solution of the Poisson equation

$$\nabla^2\mathbf{A} = -\mathbf{s}(\mathbf{x})$$

is

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\mathbf{s}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V',$$

where $\mathbf{s}(\mathbf{x})$ is a specified source. [10%]

(b) The static magnetic field, $\mathbf{B}(\mathbf{x})$, has a vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}$. This vector potential is governed by

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{J}(\mathbf{x}),$$

where μ_0 is the permeability of free space and $\mathbf{J}$ is a known distribution of current density. Use the Green's function solution for $\mathbf{A}$ to deduce the Biot-Savart law,

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{x}') \times \mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}V', \qquad \text{where} \qquad \mathbf{r} = \mathbf{x}-\mathbf{x}'.$$

Now use the vector identity

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$

to show that the Biot-Savart law ensures $\nabla \cdot \mathbf{B} = 0$. [40%]

(c) For a non-static magnetic field, the vector potential is governed by the inhomogeneous wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}(\mathbf{x}, t),$$

where c is the speed of light. Write down the generalised version of the Green's function which is appropriate to this time-dependent problem. Explain the physical origin of this generalisation. Why is this generalised version of the Green's function also important in acoustics? [20%]

2 (a) Outline the difference between dispersive and non-dispersive waves, and provide a physical interpretation of d'Alembert's solution to the one-dimensional wave equation. Why do dispersive waves not admit d'Alembert's solution? [20%]

(b) A one-dimensional, slowly-modulated wave packet contains a narrow band of wavenumbers centred around the mean value k_0. It can be expressed as

$$\varphi(x,t) = \int a(k) \exp\left[i(kx - \varpi t)\right] \mathrm{d}k,$$

where $a(k)$ is sharply peaked at k_0 and the dispersion relationship is $\varpi = \varpi(k)$.

(i) Explain why it is legitimate to use the local approximation

$$\varpi(k) = \varpi_0 + (k - k_0)\left(\frac{\mathrm{d}\varpi}{\mathrm{d}k}\right)_0$$

in the above integral for $\varphi(x,t)$. [5%]

(ii) Use this local approximation to show that $\varphi(x,t)$ can be written as

$$\varphi(x,t) = A \exp\left[i(k_0 x - \varpi_0 t)\right],$$

where the slowly-modulated amplitude function has the form $A = A(x - (\mathrm{d}\varpi/\mathrm{d}k)_0 t)$, and hence deduce an expression for the group velocity of the wave packet. [35%]

(c) In an electromagnetic waveguide the axial component of the magnetic field, B_z, takes the form

$$B_z = f(x,y) \cos(kz - \varpi t).$$

The dispersion relationship is

$$\varpi^2 = \varpi_{\mathrm{min}}^2 + c^2 k^2,$$

c being the speed of light and ϖ_{min} a cut-off frequency.

(i) Find expressions for the phase and group velocities, c_p and c_g, as functions of c, ϖ and ϖ_{min}. Hence show that $c_p c_g = c^2$. [20%]

(ii) A wave packet of length L and dominant wavenumber k_0 passes along the waveguide. Sketch the wave pattern, distinguishing between the speed of the packet and the speed of propagation of the maxima of B_z. Indicate where the maxima of B_z are born and die. [20%]

3 A surface is given by $z(r,\theta) = r\cot\alpha$, where (r,θ,z) are cylindrical coordinates and $0 < \alpha < \pi/2$ is a constant.

(a) Sketch the surface, and give a geometrical interpretation of α. [10%]

(b) Find an expression for the length of the path along the surface described by the function $r = f(\theta)$, starting from θ_1 and finishing at θ_2. [20%]

(c) Using the Beltrami identity, or otherwise, show that f will extremise the path length if the quantity [30%]

$$\frac{f(\theta)^2}{\sqrt{f(\theta)^2 + \csc^2(\alpha) f'(\theta)^2}}$$

is constant along the path.

(d) Find and sketch the shortest path along the surface from $(r,\theta) = (r_0, -\beta)$ to (r_0, β), and show that the minimum value of r along this path is

$$r_{min} = r_0 \cos(\beta \sin(\alpha)).$$

[40%]

Hint 1: You may quote the result that the Euler-Lagrange equation $\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0$ is equivalent to

$$-\frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0.$$

Hint 2: You may use the standard integral

$$\int \frac{1}{\sqrt{(x^4/a^2) - x^2}}\mathrm{d}x = \sec^{-1}(x/a).$$

4 (a) Consider the following differential equation for $u(x)$, which is to be solved for $0 < x < 1$,

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \frac{2}{x}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = 0, \qquad \begin{array}{l} u(0) = 0,\ u'(0) = 0, \\ u(1) = 0,\ u'(1) = 0. \end{array}$$

(i) Find a weak form of the equation, and explain why it is not possible to deduce a variational form. [20%]

(ii) Multiply the original equation by x, find the new weak form, and hence find an equivalent variational form. [20%]

(b) Anna has recently completed her engineering degree, and is making financial plans. She starts her career with no savings, $S(0) = 0$, but in forty years time she will need $S(40) = S_F$ to retire comfortably. Anna secures a good job, which provides a constant income I. She also invests her savings, S, in an account paying an interest rate r, (generating an additional income rS) and consumes (spends) at a rate $C(t)$, such that, overall, her savings grow as

$$\dot{S} = rS + I - C.$$

(i) Show that Anna's savings target is equivalent to the integral constraint [20%]

$$\int_0^{40} \mathrm{e}^{r(40-t)}(I - C(t))\mathrm{d}t = S_F.$$

(ii) Anna wishes to pace her consumption (spending), $C(t)$, over her career in order to maximize her utility while still hitting her retirement savings target. To do this, she decides to maximize the quantity

$$U = \int_0^{40} \ln{(C/C_0)}\mathrm{d}t,$$

where C_0 is a (constant) level of spending required for basic subsistence. Show that, to maximize U while meeting her retirement savings target, Anna should choose $C(t)$ of the form

$$C(t) = \frac{1}{\lambda}\mathrm{e}^{r(t-40)},$$

and find an expression for the constant λ. [40%]

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