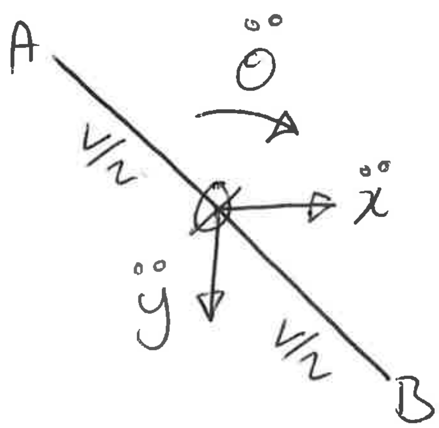
ENGINEERING PART IB – PAPER 1 MECHANICS

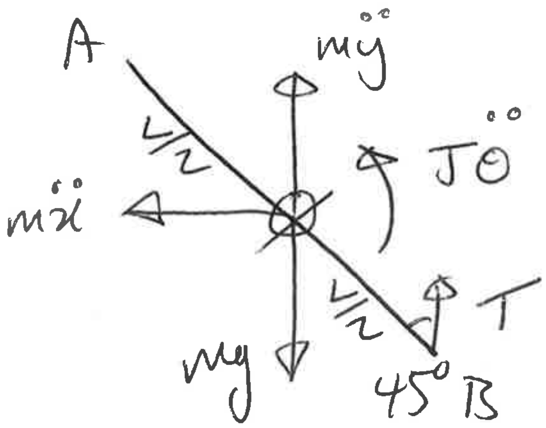
2015 D J COLE

1 a)

accns



forces



resolve forces horizontally

$$m\ddot{x} = 0$$

$$\underline{\underline{\ddot{x} = 0}}$$

b) sum forces vertically ↑+

$$m\ddot{y} + T = mg \quad \text{——— (1)}$$

moments about centre of mass

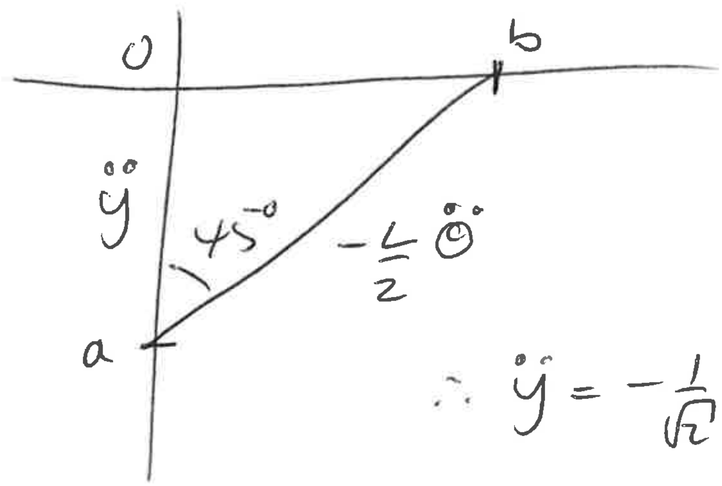
$$J\ddot{\theta} + \frac{T}{\sqrt{2}} \cdot \frac{L}{2} = 0$$

$$T = -J\ddot{\theta}\,\frac{2\sqrt{2}}{L} \quad \text{—— (2)}$$

substitute into (1)

$$m\ddot{y} - J\ddot{\theta}\,\frac{2\sqrt{2}}{L} = mg. \quad \text{— (3)}$$

acceleration diagram to relate $\ddot{y}$ and $\ddot{\theta}$



$$\therefore \ddot{y} = -\frac{1}{\sqrt{2}} \cdot \frac{L}{2}\ddot{\theta} \quad -(4)$$

subst (4) into (3)

$$-m\frac{L}{2\sqrt{2}}\ddot{\theta} - J\ddot{\theta}\,\frac{2\sqrt{2}}{L} = mg \qquad J = \frac{1}{12}mL^2$$

$$-\ddot{\theta}\left(\frac{L}{2\sqrt{2}}\cancel{m} + \frac{1}{12}\cancel{m}L^2\frac{2\sqrt{2}}{L}\right) = \cancel{m}g$$

$$\ddot{\theta} = \frac{-g}{L\left(\frac{1}{2\sqrt{2}} + \frac{2\sqrt{2}}{12}\right)}$$

(i) $$\ddot{\theta} = -\frac{g}{L} \cdot \frac{6\sqrt{2}}{5}$$

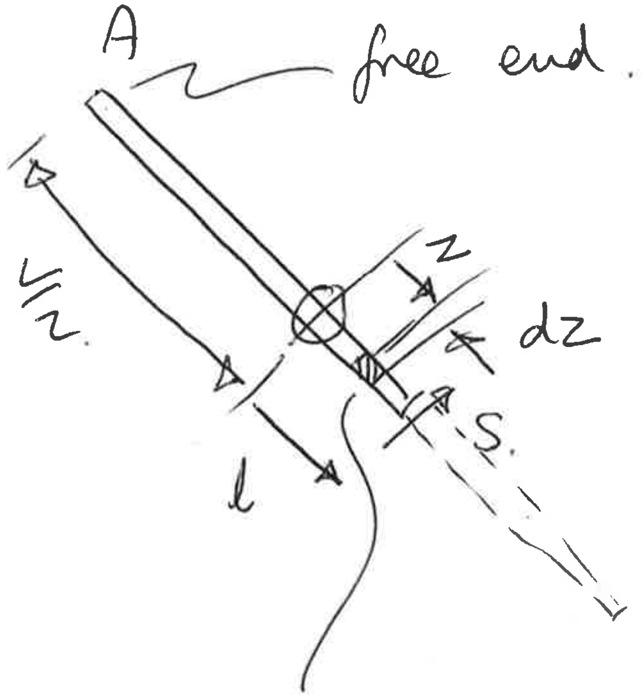
(ii) from (4) $$\ddot{y} = \frac{\cancel{L}}{2\cancel{\sqrt{2}}} \cdot \frac{g}{\cancel{L}}\,\frac{{}^{3}\cancel{6}\cancel{\sqrt{2}}}{5} = \frac{3}{5}g$$

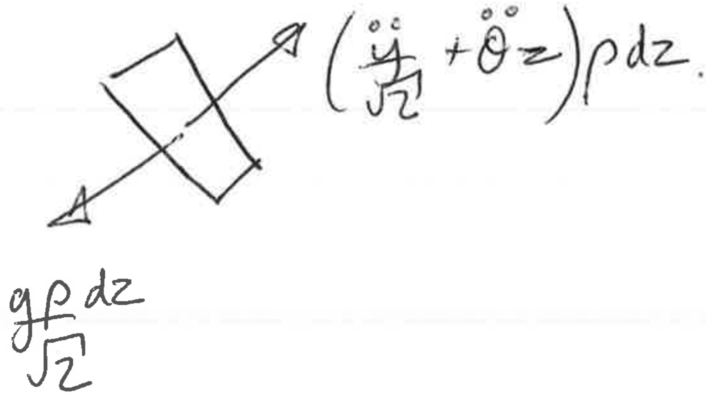
(iii) from (1) $$m\ddot{y} + T = mg.$$

$$m\frac{3}{5}g + T = mg$$

$$T = \frac{2}{5}mg = \frac{2}{5}L\rho g$$

c)





$$w(z)\swarrow = \frac{g}{\sqrt{2}}\rho - \left(\frac{\ddot{y}}{\sqrt{2}} + \ddot{\theta} z\right)\rho$$

$$= \rho\left(\frac{g}{\sqrt{2}} - \frac{3}{5\sqrt{2}}g + \frac{g}{L}\frac{6\sqrt{2}}{5}z\right)$$

$$= \rho g L\left(\frac{2}{5\sqrt{2}L} + \frac{z}{L^2}\frac{6\sqrt{2}}{5}\right)$$

$$= \frac{\sqrt{2}}{5}\rho g L\left(\frac{1}{L} + \frac{6z}{L^2}\right)$$

Shear force S

$$S(l) - \int_{-\frac{L}{2}}^{l} w(z)\,dz = 0$$

$$S(l) = \int_{-\frac{L}{2}}^{l} \frac{\sqrt{2}}{5}\rho g L\left(\frac{1}{L} + \frac{6z}{L^2}\right)dz$$

$$= \frac{\sqrt{2}}{5}\rho g L\left[\frac{z}{L} + \frac{6z^2}{2L^2}\right]_{-\frac{L}{2}}^{l}$$

$$= \frac{\sqrt{2}}{5}\rho g L\left(\left(\frac{l}{L} + \frac{3l^2}{L^2}\right) - \left(-\frac{L}{2L} + \frac{3L^2}{L^2 4}\right)\right)$$

$$= \frac{\sqrt{2}}{5}\rho g L\left(\frac{3l^2}{L^2} + \frac{l}{L} - \frac{1}{4}\right)$$

check $S(l) = 0$ when $l = -\frac{L}{2}$ ✓.

$S(l) = \frac{\sqrt{2}}{5}\rho g L$ when $l = \frac{L}{2}$ ✓. $\left(= \frac{T}{\sqrt{2}}\right)$

Max BM when $S(l) = 0$.

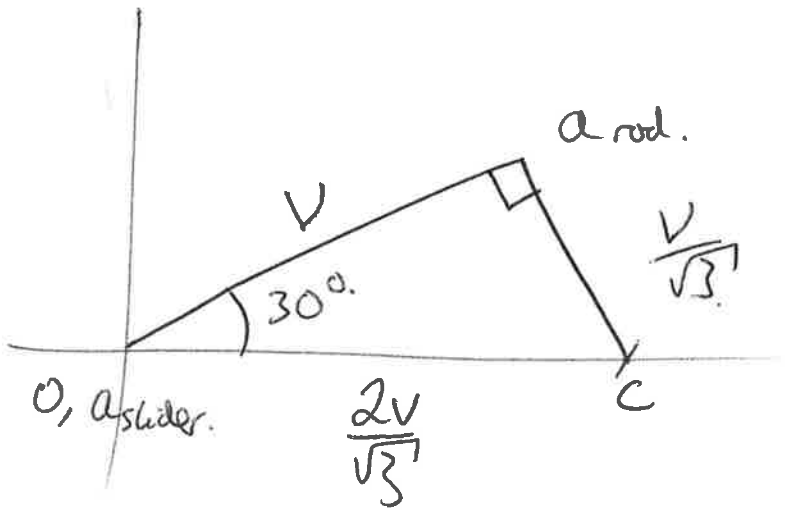
$$\frac{3l^2}{L^2} + \frac{l}{L} - \frac{1}{4} = 0.$$

$$\frac{l}{L} = \frac{-1 \pm \sqrt{1 + 4.3.\frac{1}{4}}}{2.3}$$

$$= -\frac{1}{6} \pm \frac{1}{3} = -\frac{1}{2} \text{ or } +\frac{1}{6}.$$

$\therefore$ BM_{max} at $\frac{L}{2} + \frac{L}{6} = \frac{2}{3}L$ from A.

2. a) velocity diagram.

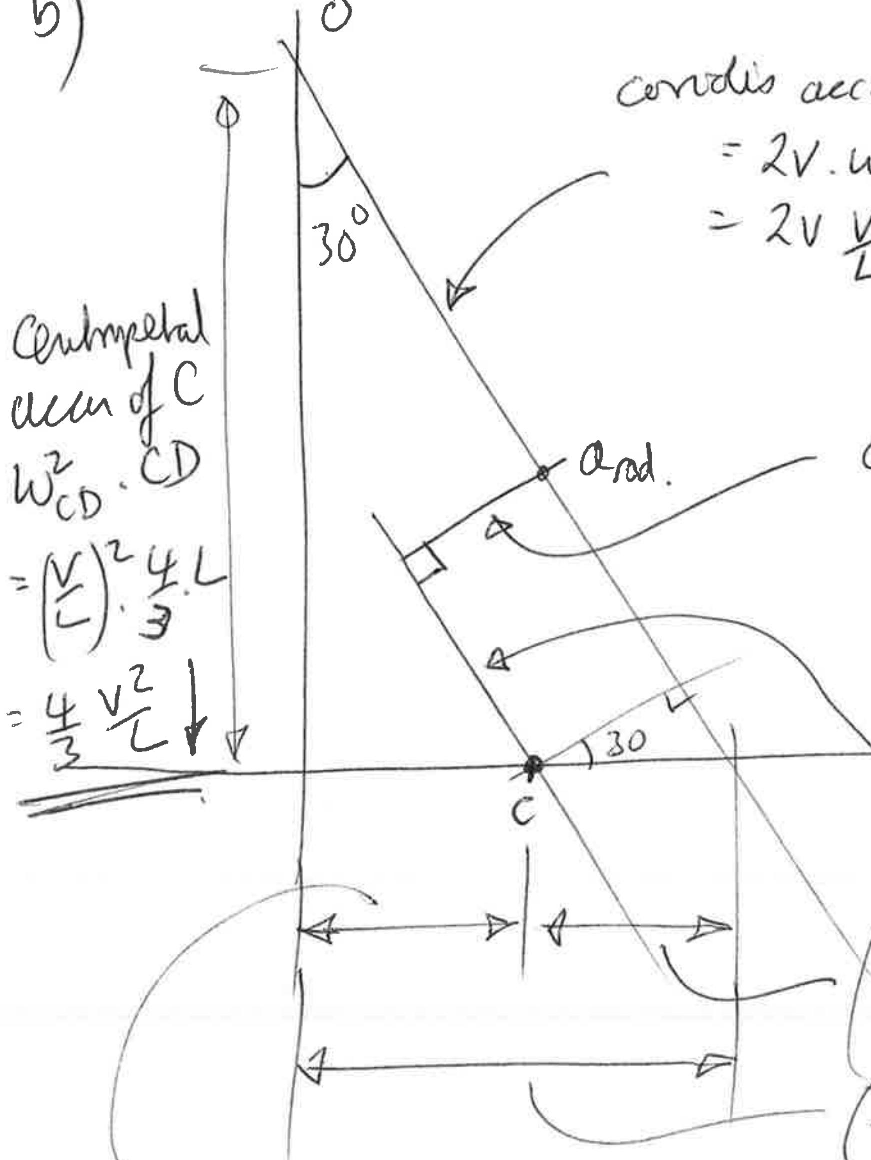


i) $V_c = \frac{2V}{\sqrt{3}} \longrightarrow$

ii) $\omega_{AC} = \frac{V_{C\,A\,rod}}{AC} = \frac{V/\sqrt{3}}{2L} = \frac{V}{L}\frac{1}{2\sqrt{3}}$

$\omega_{CD} = \frac{V_c}{CD} = \frac{2V/\sqrt{3}}{L} = \frac{V}{L}\frac{2}{\sqrt{3}}$

b)



coriolis accn of C.

$= 2V \cdot \omega_{AC}$

$= 2V \frac{V}{L} \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \frac{V^2}{L}$

Centripetal accn of C

$\omega_{CD}^2 \cdot CD$

$= \left(\frac{V}{L}\right)^2 \cdot \frac{4}{3} L$

$= \frac{4}{3} \frac{V^2}{L} \downarrow$

centripetal accn of C

$= \omega_{AC}^2 \, AC$

$= \frac{V^2}{L^2} \cdot \frac{1}{12} \cdot 2L = \frac{V^2}{L} \cdot \frac{1}{6}$

tangential accn of C.

$\left(\frac{V^2}{L} \cdot \frac{1}{6}\right) \frac{2}{\sqrt{3}}$

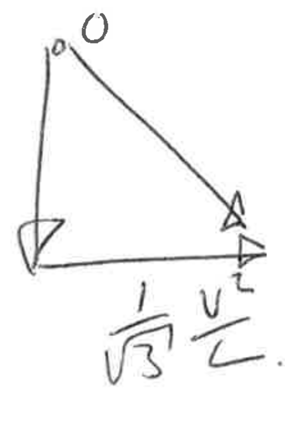
$\left(\frac{4}{3} \frac{V^2}{L}\right) \cdot \frac{1}{\sqrt{3}}$

tangential accn of C $\dot{\omega}_{CD} \cdot CD$

$= \left(\frac{4}{3} \cdot \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right) \frac{V^2}{L} = \frac{1}{\sqrt{3}} \frac{V^2}{L} \rightarrow$

angular acceleration of box $= 0$

linear accn of box



c) wheel power.

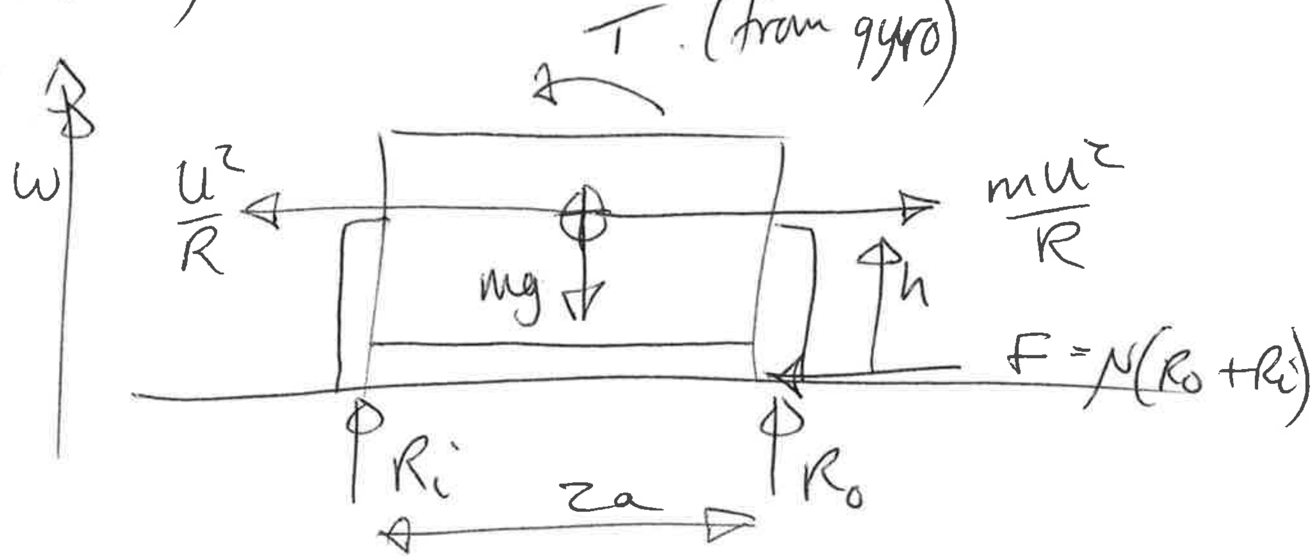
$$F_{act} \cdot V = m \cdot acc_n \cdot V_c + 4Q \cdot \omega_{CD}$$

$$F_{act} \cdot V = m \cdot \frac{1}{\sqrt{3}} \frac{V^2}{L} \cdot \frac{2V}{\sqrt{3}} + 4Q \frac{V}{L} \frac{2}{\sqrt{3}}$$

$$F_{act} = m \frac{V^2}{L} \cdot \frac{2}{3} + \frac{Q}{L} \frac{8}{\sqrt{3}}$$

Fact puts actuator in compression.

3. a)



rollover when $R_i = 0$

mts about ~~outer~~ outer tyre/road contact

$$\frac{mU^2}{R} \cdot h = mga$$

$$\frac{U^2_{rollover}}{R} = \frac{ga}{h}$$

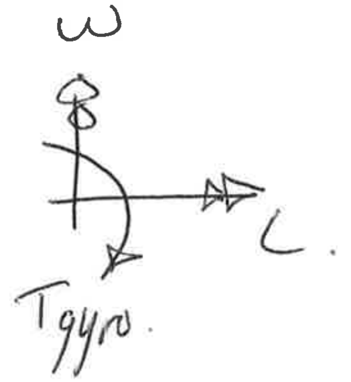
$$\underline{\underline{U_{rollover} = \sqrt{\frac{Rga}{h}}}}$$

b) if $R_o = R_i = \frac{mg}{2}$ gyro spin

$$T_{g} = \frac{mU^2}{R} \cdot h = J\omega\Omega$$

precession $\omega = \frac{U}{R}$ $\therefore \Omega = \frac{mUh}{J}$

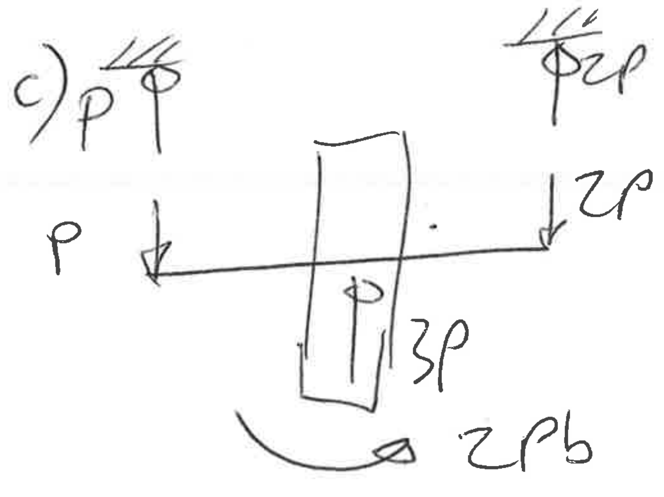
direction $\quad T_{gyro} = \underbrace{\omega}_{\text{precession}} \times \underbrace{L}_{\text{spin momentum}}$



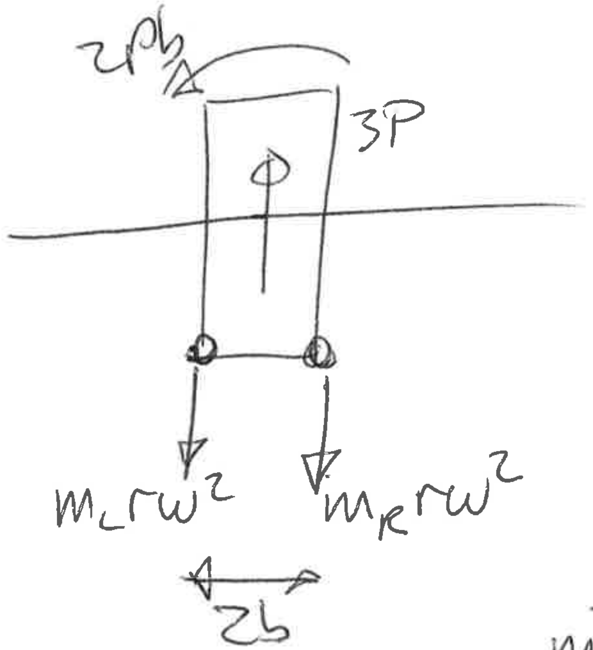
wheel rotation

hence gyro must spin in opposite direction to wheels.

c)



add masses to ~~provide~~ oppose 3P and 2Pb.



sum forces

$(m_L + m_r) r\omega^2 = 3P$

moments

$2P\cancel{b} + m_L r\omega^2 \cancel{b} = m_R r\omega^2 \cancel{b}$

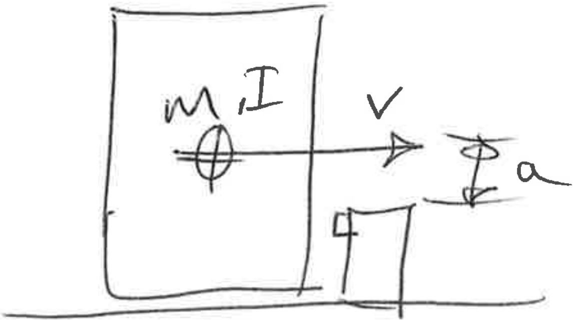
hence $P = 2m_L r\omega^2$

$m_L = \dfrac{P}{2r\omega^2}, \quad m_R = \dfrac{5}{2}\dfrac{P}{r\omega^2}$

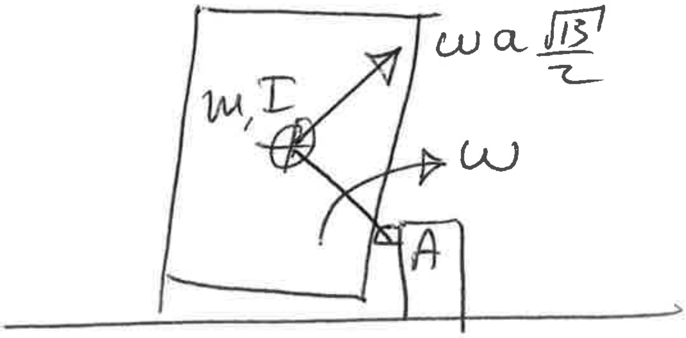
mounted on opposite side to forces on bearings.

4 a)

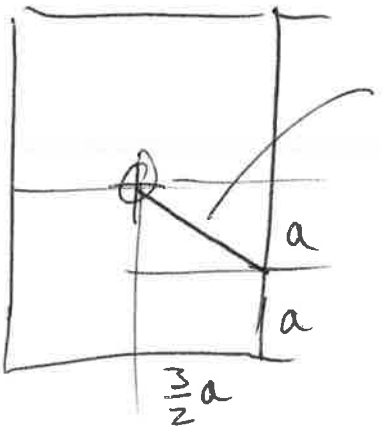
before



after



geometry



$$\sqrt{a^2 + \left(\tfrac{3}{2}a\right)^2} = a\frac{\sqrt{13}}{2}$$

$$I_G = m\left(\tfrac{1}{12}(3a)^2 + \tfrac{1}{12}(4a)^2\right)$$
$$= \frac{m}{12}\left(9a^2 + 16a^2\right)$$
$$= \frac{25}{12}ma^2$$

moment of momentum conserved about impact point

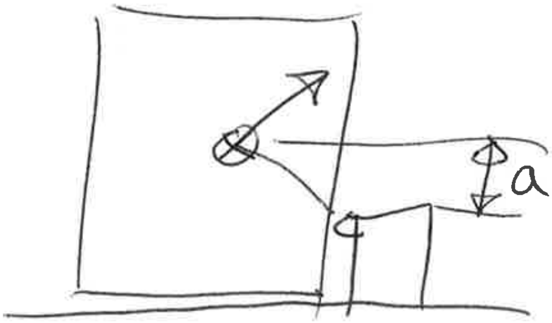
$$(mv)\,a = \left(m\omega a\frac{\sqrt{13}}{2}\right) a\frac{\sqrt{13}}{2} + \frac{25}{12}ma^2\omega$$

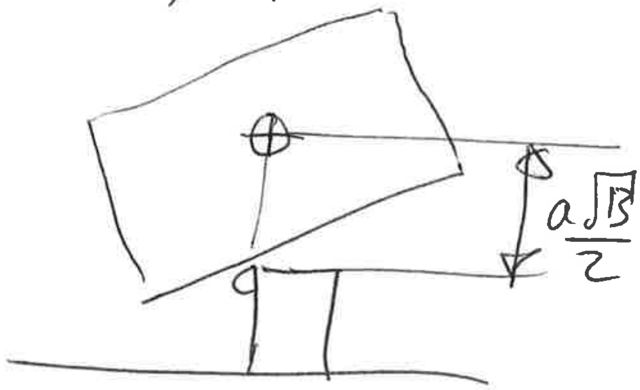
$$mva = m\omega a^2\,\frac{13}{4} + \frac{25}{12}m\omega a^2$$
$$= \frac{16}{3}m\omega a^2$$

$$\underline{\underline{\omega = \frac{3}{16}\frac{v}{a}}}$$

b) loss of KE = gain in PE

consider velocities ~~at~~ immediately after impact.





$\cancel{KE = \frac{1}{2} I\omega^2}$

$= \frac{1}{2} I\omega^2$

$$KE = \frac{1}{2}\left(\frac{25}{12}ma^2 + ma^2\frac{13}{4}\right)\omega^2 = \frac{8}{3}ma^2\omega^2$$

$$PE = mg\left(a\frac{\sqrt{13}}{2} - a\right)$$

$\cancel{\Delta KE = \Delta PE}$.

$$\frac{8}{3}ma^2\omega^2 = \frac{ma}{2}g(\sqrt{13}-2)$$

$$\omega^2 = \frac{g}{a}(\sqrt{13}-2)\frac{1}{2}\cdot\frac{3}{8}$$

$$\omega = \sqrt{\frac{g}{a}\frac{3}{16}(\sqrt{13}-2)}$$

but $v = \omega\frac{16}{3}a = \frac{16}{3}a\sqrt{\frac{g}{a}\frac{3}{16}(\sqrt{13}-2)}$

$$= \sqrt{ga\frac{16}{3}(\sqrt{13}-2)}$$

c) $4a = 3.2\,m$, $3a = 2.4\,m$, $\therefore\ a = 0.8\,m$

$M = 20.10^3\ kg.$

$$V = \sqrt{ga\,\tfrac{16}{3}\left(\sqrt{13}-2\right)} = \sqrt{9.81 \cdot 0.8 \cdot \tfrac{16}{3}\left(\sqrt{13}-2\right)}$$

$$= \underline{\underline{8.194\ m/s}}$$

energy absorbed

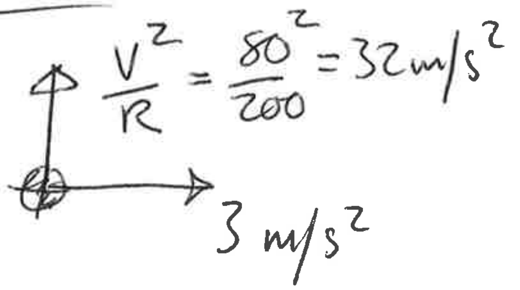
$$\tfrac{1}{2}mv^2 - mgh \qquad h = a\left(\tfrac{\sqrt{13}}{2} - 1\right)$$

$$= \tfrac{1}{2} \cdot 20.10^3 \cdot 8.194^2 - 20.10^3 \cdot 9.81 \cdot 0.8\left(\tfrac{\sqrt{13}}{2} - 1\right)$$

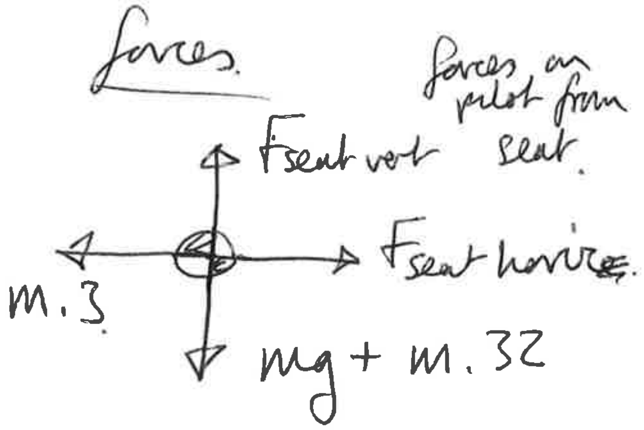
$$= \underline{\underline{545.4\ kJ}}$$

5 a) free body diag of pilot.

accns.

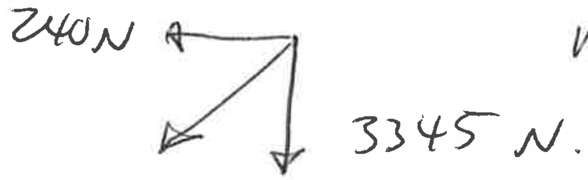


forces.



$$F_{seat\ vert} = mg + m.32$$
$$= 80(9.81 + 32)$$
$$= 3344.8 \, N.$$
$$F_{seat\ horiz} = m.3 = 80.3 = 240 \, N.$$

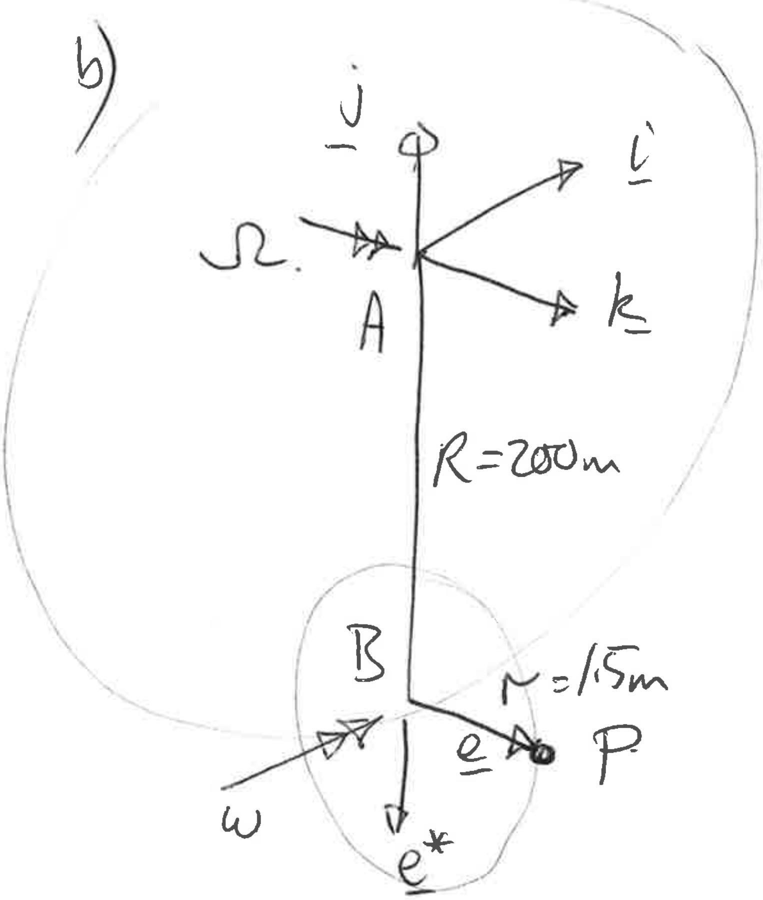
∴ force on seat from pilot:



magnitude = $\underline{\underline{3353.4 \, N}}$

~~b)~~

b)



$\underline{i}, \underline{j}$ rotate at $\Omega \underline{k}$

$\underline{e}, \underline{e}^*$ rotate at $\Omega \underline{k} + \omega \underline{i}$

$\Omega = \frac{80 m/s}{200 m} = \frac{2}{5} = 0.4 \text{ rad/s}$

$\dot{\Omega} = \frac{3 m/s^2}{200 m} = 0.015 \text{ rad/s}^2$

$\omega = 3342 \text{ rpm}$
$= 350 \text{ rad/s}$

$$\underline{R}_P = -R\underline{j} + r\underline{e}$$

$$\dot{\underline{R}}_P = -R(\Omega \underline{k} \times \underline{j}) + r(\Omega \underline{k} + \omega \underline{i}) \times \underline{e}$$

$$= \Omega R \underline{i} + \underbrace{\Omega r \underline{k} \times \underline{e}}_{\text{instantaneously zero}} + r\omega \underline{e}^*$$

$\underline{e}^* = \underline{i} \times \underline{e} = -\underline{j}$

$$\therefore \dot{\underline{R}}_P = 0.4 \cdot 200 \underline{i} - 1.5 \times 350 \cdot \underline{j}$$

$$\dot{\underline{R}}_P = 80 \underline{i} - 525 \underline{j} \text{ m/s}$$

c) $\ddot{\underline{R}}_p = \Omega R(\Omega \underline{k} \times \underline{i}) + \dot{\Omega} R \underline{i}$

$+ \Omega r(\cancel{\dot{\underline{k}} \times \underline{e}}^{\,0} + \underline{k} \times \dot{\underline{e}})$

$+ \cancel{\dot{\Omega} r \underline{k} \times \underline{e}}^{\,0}$

$+ r\omega(\Omega k + \omega \underline{i}) \times \underline{e}^*$

$= \Omega^2 R \underline{j} + \dot{\Omega} R \underline{i} + \Omega r(\underline{k} \times (\cancel{\Omega \underline{k}}^{\,\underline{k} \times \underline{k} = 0.} + \omega \underline{i}) \times \underline{e})$

$+ r\omega\Omega(\underline{k} \times \underline{e}^*) + r\omega^2(\underline{i} \times \underline{e}^*)$

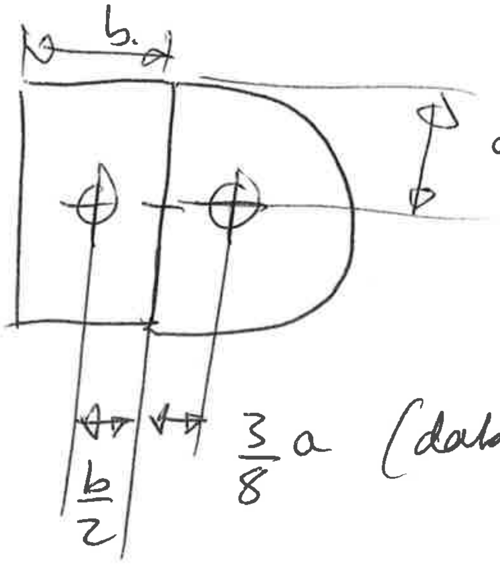
$= \Omega^2 R \underline{j} + \omega \Omega r(\underline{k} \times \underbrace{\underline{i} \times \underline{e}}_{-\underline{j}}) + r\omega\Omega \underline{i} - r\omega^2 \underline{k}$
$+ \dot{\Omega} R \underline{i}$

$= \Omega^2 R \underline{j} + \dot{\Omega} R \underline{i} + 2 r\omega\Omega \underline{i} - r\omega^2 \underline{k}$

$= 0.4^2 . 200 \underline{j} + (3 + 2 . 0.4 . 350 . 1.5) \underline{i} - 1.5 . 350^2 \underline{k}$

$= 423 \underline{i} + 32 \underline{j} - 183750 \underline{k} \quad m/s^2$

6 a) combined centre of mass should be at centre of spherical surface.



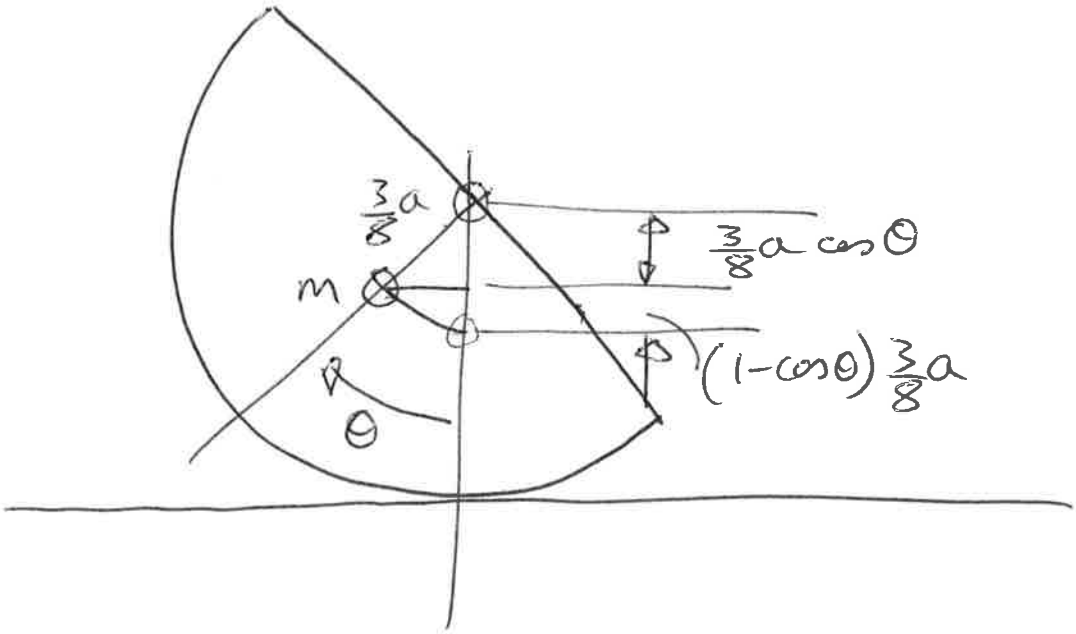
vol of hemisphere $\frac{2}{3}\pi a^3$

$$\left(\pi a^2 b \rho g\right)\frac{b}{2} = \left(\frac{2\pi a^3}{3}\rho g\right)\frac{3}{8}a$$

$$b^2 = \frac{1}{2}a^2$$

$$\underline{\underline{b = \frac{a}{\sqrt{2}}}}$$

b)



$$PE = mg(1-\cos\theta)\frac{3}{8}a \quad \text{(hemisphere)}$$

$$KE = \underbrace{\frac{1}{2}m(a\dot{\theta})^2}_{\text{pellet}} + \underbrace{\frac{1}{2}m\left(\frac{5}{8}a\dot{\theta}\right)^2 + \frac{1}{2}J\dot{\theta}^2}_{\text{hemisphere}}$$

$$= \frac{1}{2}m\dot{\theta}^2\left(a^2 + \left(\frac{5}{8}a\right)^2 + \frac{83}{320}a^2\right)$$

$$= \frac{1}{2}\dot{\theta}^2 ma^2\left(1 + \frac{25}{64} + \frac{83}{320}\right)$$

$$= \frac{1}{2}\dot{\theta}^2 ma^2\left(\frac{320 + 125 + 83}{320}\right)$$

$$KE = \frac{1}{2}\dot{\theta}^2 ma^2\,\frac{33}{20}.$$

$$PE + KE = \text{const.}$$

$$mg(1-\cos\theta)\frac{3}{8}a + \frac{1}{2}\dot{\theta}^2 ma^2 \frac{33}{20} = \text{const.}$$

differentiate with respect to θ $\left(\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = \ddot{\theta}\right)$

$$\frac{3}{8}mga\sin\theta + \ddot{\theta}ma^2\frac{33}{20} = 0.$$

small angle $\sin\theta \to \theta$.

$$\ddot{\theta}ma^2\frac{33}{20} + \frac{3}{8}mga\,\theta = 0.$$

$$\ddot{\theta} = -\frac{5}{22}\frac{g}{a}\theta$$

$$\ddot{\theta} = -\omega_n^2\theta$$

$$\therefore \quad \omega_n = \sqrt{\frac{5}{22}\frac{g}{a}}$$

$$= \sqrt{\frac{5}{22}\frac{9.81}{0.02}}$$

$$\underline{\underline{\omega_n = 10.56 \text{ rad/s}}}$$