1 (a)

(*) If Real $[s(k)]<0$ for all k the system is stable.

The essence of the approach is as follows:

Small amplitude perturbations (marked below with a prime) are introduced about the pressure, velocity, etc., of the steady base flow (say $\underline{u}_0, p_0$) so that

$$\underline{u} = \underline{u}_0 + \underline{u}'(x, y, z, t)$$
$$p = p_0 + p'(x, y, z, t)$$

and substituted into the governing equations of motion & boundary conditions. This system is then linearised, i.e. products of small terms are neglected (as diminishingly small). A given disturbance to the base flow can be Fourier analysed spatially & expressed as an integral sum of normal modes over a range of wavenumbers k. Owing to there being an absence of terms in the governing equations involving products of perturbations, we can solve for the growth rate $s(k)$ by taking a single mode for which k is treated as a parameter - subsequently sweeping through all values of k. Solutions to linearised system sought in terms of normal mode solutions, eg $p' = \hat{p}(z)e^{ikx+st}$ (*)

For this specific problem, given flow is inviscid, governing eq's are

$$\nabla^2\phi_u = \nabla^2\phi_l = 0$$

and boundary conditions

$$\frac{\partial\phi_u}{\partial r} = \frac{\partial\phi_l}{\partial r} = 0 \quad \text{on } r=a.$$

(b). Given $s^2 = \frac{k\gamma}{\rho_l+\rho_u}\left\{ g\frac{(\rho_u-\rho_l)}{\gamma} - k^2 \right\}$ as the growth rate then we have immediately $s^2>0$ provided $k^2 < g\frac{(\rho_u-\rho_l)}{\gamma}$, equivalently

$$(ka)^2 < g\frac{(\rho_u-\rho_l)}{\gamma}a^2 \quad \text{for instability.} \qquad \text{(bI)}$$

dimensionless wave no.

1 (c) Note that the quantity $G = \frac{g(\rho_u - \rho_L)a^2}{\gamma} \equiv \frac{g(\rho_u-\rho_L)}{\gamma/a^2}$ ← destabilising effects of buoyancy ← stabilising effects of interfacial tension

is dimensionless.*

It is the key governing parameter of the system & characterises the relative effects of destabilising density difference to stablising surface tension.

$$* \left[\frac{g(\rho_u-\rho_L)a^2}{\gamma} \right] \sim \left[\frac{L}{T^2} \cdot \frac{M}{L^3} \cdot L^2 \cdot \frac{1}{MLT^{-2}/L} \right] = 1 \Rightarrow \text{dimensionless.}$$

Thus for sufficiently large G we expect the system to be unstable. Indeed, the result from part (b) indicates that as density differences increase so a wider range of dimensionless radial wavenumbers perturbations result in instability. By contrast, increasing surface tension {reduces RHS of eq^n (b1)} may be regarded as stabilising large wavenumber perturbations, etc.

(d) In experiment, one would expect to observe the fastest growing mode. In dimensionless form, the growth rate

$$s^2 \Big/ \frac{(ka)\gamma}{(\rho_L+\rho_u)a^3} = \underbrace{\frac{g(\rho_u-\rho_L)a^2}{\gamma}}_{G} - (ka)^2 \qquad \text{—— (d1)}$$

so that the most unstable mode depends on the value of G.

We are given that:

	ka
with $n=0$	3.83, - - -
with $n=1$	1.84, - - -
with $n=2$	3.05, - - -

← Given

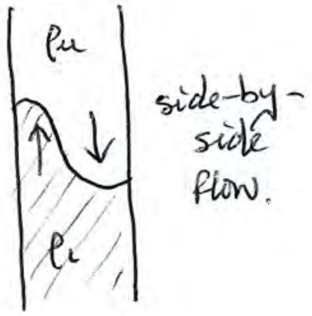
i.e. axisymmetric mode

non-axisymmetric

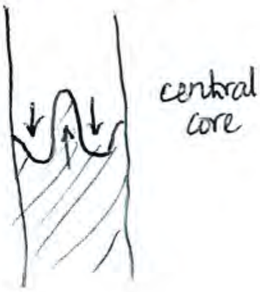
PTO

1 (d) contd.

For small values of G, i.e. small density differences, the RHS of (d1) is largest, not with $n=0$ but with $n=1$ {not ka is smaller with $n=1$ than $n=0$, $n=2$}. Thus for small density differences, the most unstable mode is <u>non-axisymmetric</u>.

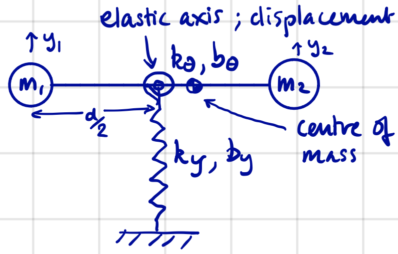


For larger values of G, the most unstable mode can correspond to a larger value of ka {growth rates will be bounded}, i.e. those for $n=0$. At larger density differences the flow observed can then be <u>axisymmetric</u>.



∴ We conclude that the results of the stability analysis are consistent with the observations.

2 (a)



Considering forces in the vertical direction,

$$-ky_1 - b\dot{y}_1 - ky_2 - b\dot{y}_2 = -k_y\left(\frac{y_1+y_2}{2}\right) - b_y\left(\frac{\dot{y}_1+\dot{y}_2}{2}\right)$$

$\Rightarrow \; 2k(y_1+y_2) = k_y(y_1+y_2)$ and $2b(\dot{y}_1+\dot{y}_2) = b_y(\dot{y}_1+\dot{y}_2)$ — separating forces proportional to displacement from those proportional to velocity.

$\Rightarrow \; k_y = 2k$ and $b_y = 2b$.

Similarly, considering moments about the elastic axis:

$$ky_1\frac{d}{2} + b\dot{y}_1\frac{d}{2} - ky_2\frac{d}{2} - b\dot{y}_2\frac{d}{2} = -k_\theta\theta - b_\theta\dot{\theta} \simeq -k_\theta\left(\frac{y_2-y_1}{d}\right) - b_\theta\left(\frac{\dot{y}_2-\dot{y}_1}{d}\right) \text{ for small } \theta$$

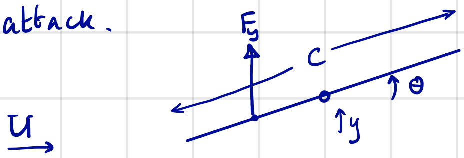
$\Rightarrow \; k\frac{d}{2}(y_1-y_2) = k_\theta\frac{(y_1-y_2)}{d} \;\Rightarrow\; k_\theta = kd^2/2$

and similarly for b $\longrightarrow$ $b_\theta = bd^2/2$

Most students answered this well. A common error was to write that $I_\theta = (m_1+m_2)d/2$

The moment of inertia about the elastic axis is $I_\theta = (m_1+m_2)\,d^2/4$

(b) We make the quasi-steady assumption: that the lift & drag forces at the instantaneous apparent angle of attack are the same as those in a steady flow at that angle of attack.



- the velocity of the leading edge is $\dot{y} - c\dot{\theta}/2$
- the apparent angle of attack is $\dfrac{-\dot{y} + c\dot{\theta}/2}{U} = \alpha$
- the equilibrium angle of attack is zero

We can express F_y in terms of the lift coefficient C_L: $F_y = \frac{1}{2}\rho U^2 c\, C_L$ per unit depth into page.

For small oscillations we can write $C_L \simeq \alpha\, \partial C_L/\partial\alpha|_0 \Rightarrow F_y = \frac{1}{2}\rho U^2 c\,\alpha \left.\frac{\partial C_L}{\partial \alpha}\right|_0$ ← angle of attack.

$$\Rightarrow \; F_y = \frac{1}{2}\rho U^2 c\,\frac{\partial C_L}{\partial\alpha}\left(\frac{-\dot{y} + c\dot{\theta}/2}{U}\right)$$ with F_y defined positive upwards.

$$F_\theta = -\frac{c}{4}F_y$$

Few students answered this correctly. Many forgot to include the chord, c. Some did not make the approximation $C_L \simeq \alpha\, \partial C_L/\partial\alpha|_0$

(c) Substitute F_y and F_θ into the translational and torsional equations of motion. Assume a modal decomposition of the form $y = Y_0 e^{st}$ and $\theta = \Theta_0 e^{st}$ and substitute into the equations of motion. Express in matrix form as:

$$\begin{bmatrix} a_1(s) & b(s) \\ c(s) & a_2(s) \end{bmatrix}\begin{bmatrix} Y_0 \\ \Theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and find s for which the determinant of the matrix is zero. (Equivalently, state that we solve the two simultaneous equations.)

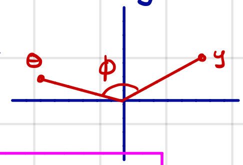
Well answered by nearly all students

This leads to a quartic: $C_0 s^4 + C_1 s^3 + C_2 s^2 + C_3 s + C_4 = 0$ with four solutions.

At least one solution has positive real part (unstable) when $C_1 C_2 C_3 < C_0 C_3^2 + C_4 C_1^2$.

This gives an algebraic expression for the parameter values at which the system is unstable.

(d) Each value of s (each eigenvalue) has a corresponding eigenfunction $\begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$, which is in general complex. $y = y_0 e^{st}$ and $\theta = \theta_0 e^{st}$ can be represented in the complex plane as:



In modulus/argument form this is $y = |y_0| e^{\mathrm{Arg}(y_0)+st}$; $\theta = |\theta_0| e^{\mathrm{Arg}(\theta_0)+st}$

The phase, ϕ, is $\mathrm{Arg}(y_0) - \mathrm{Arg}(\theta_0) = \mathrm{Arg}\left(\frac{y_0}{\theta_0}\right)$

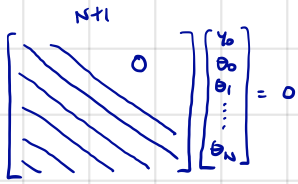
Around 1/3 students showed that they understood how to find the phase, and that the phase information is contained in y_0 & θ_0

The instantaneous power extracted from the wind is $F_y \dot{y} + F_\theta \dot{\theta}$. This can be integrated over a cycle, which has period $T = 2\pi/s_i$, where $s_i = \mathrm{Imag}(s)$: $W = \int_0^T (F_y \dot{y} + F_\theta \dot{\theta})\,dt$

Around 1/2 students gave reasonable answers

(e) In the series of identical canopies, the analysis can proceed in a few different directions. Each canopy adds a single extra degree of freedom. For N canopies the corresponding matrix is $(N+1)\times(N+1)$:

e.g. if we express the state as the position of the first canopy and the angles with all the others then the matrix is →



We can solve this in the same way as before but the calculation will be expensive when N is large.

When the number of canopies is finite, there is a start and and end canopy, meaning that the steady state is a standing wave. When the number of canopies is infinite, travelling waves can also exist. [For this, it would be better to formulate it as a continuous problem and use local stability analysis].

Almost all students saw that this would lead to an eigenvalue problem for a large matrix.

Around 1/4 of the students correctly identified that the infinite limit becomes like a continuous system with no pinned boundaries and wave reflections.

Matthew Juniper
Jan 2019

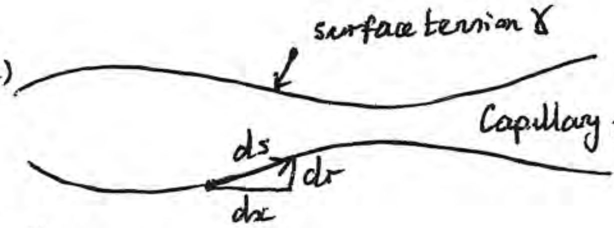
3 (a). Given $r = \alpha + \beta\cos kx$

Use vol. cons. to relate α to β: $$\underbrace{\pi a^2 \lambda}_{\substack{\text{vol. undisturbed} \\ \text{jet in length } \lambda}} = \int_0^\lambda \pi r^2\,dx = \underbrace{\pi \int_0^\lambda (\alpha + \beta\cos kx)^2\,dx}_{\substack{\text{vol. perturbed jet} \\ \text{in length } \lambda}}$$

$$\therefore \quad \frac{\pi}{\pi} a^2 \lambda = \int_0^\lambda \alpha^2 + 2\alpha\beta\cos kx + \beta^2 \cos^2 kx\,dx.$$

$$\Rightarrow \quad a^2 = \alpha^2 + \frac{\beta^2}{2}, \quad \text{i.e.} \quad \alpha = a\left(1 - \frac{1}{2}\frac{\beta^2}{a^2}\right)^{1/2}. \qquad -(1)$$

$$PE_{jet} = \gamma \times (\text{surface area}) \qquad -(2)$$



Surface area disturbed jet in length λ

$$SA_1 = \int_{x=0}^{\lambda} 2\pi r\,ds \qquad -(3)$$

Given $ds^2 = dx^2 + dr^2 = dx^2\left(1 + \frac{dr^2}{dx^2}\right) \Rightarrow ds = dx\left(1 + \left(\frac{dr}{dx}\right)^2\right)^{1/2}$

i.e. $ds = dx\left(1 + \frac{1}{2}\left(\frac{dr}{dx}\right)^2 + \dots\right)$ and $\frac{dr}{dx} = -\beta k \sin kx$

so that $ds \approx \left(1 + \frac{1}{2}\beta^2 k^2 \sin^2 kx\right)dx$

Hence,

$$SA_1 = 2\pi \int_0^\lambda (\alpha + \beta\cos kx)\left(1 + \frac{1}{2}\beta^2 k^2 \sin^2 kx\right)dx$$

$$SA_1 = 2\pi\alpha\lambda\left[1 + \left(\frac{\beta k}{2}\right)^2\right]$$

$$\Rightarrow \quad PE_{disturb.\,jet} = \gamma\, 2\pi\alpha\lambda\left(1 + \left(\frac{\beta k}{2}\right)^2\right)$$

Surface area undisturbed jet in length λ $= 2\pi a \lambda$

$$\Rightarrow \quad PE_{undisturbed\,jet} = \gamma\, 2\pi a \lambda$$

Hence, $$\frac{PE_{disturbed}}{PE_{undisturbed}} = \frac{\alpha}{a}\left[1 + \frac{\beta^2 k^2}{4}\right].$$ Now eliminate α.

$-(4)$

Question 3

(b) Consider a ring of fluid • radius r_1, circumferential velocity u_1
that is displaced outwards to • radius r_2, with circumferential velocity u_2'

Neglecting viscous forces $r_1 u_1 = r_2 u_2'$ as angular mom. conserved

$$\Rightarrow u_2' = \left(\frac{r_1}{r_2}\right) u_1$$

As $-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r}$, the pressure gradient is just sufficient to hold a ring with velocity u_2 at the radius r_2, thus

if $\frac{u_2'^2}{r_2^2} > \frac{u_2^2}{r_2^2}$, i.e. $u_2'^2 > u_2^2$ then radial press. grad. is not sufficient to offset the centrifugal force & ring continues outwards (unstable)

Thus require $u_2'^2 \leq u_2^2$ for stability.

Sub. for $u_2' = (r_1/r_2)u_1$ gives

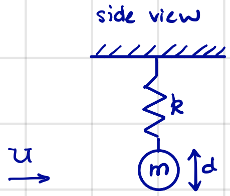
$$r_1^2 u_1^2 \leq r_2^2 u_2^2 \quad \Rightarrow \quad r_2^2 u_2^2 - r_1^2 u_2^2 \geq 0 \qquad (r_1 > r_2)$$

$$\text{i.e.} \quad \frac{d}{dr}(r^2 u^2) \geq 0$$

now $u = r\Omega$

$$\Rightarrow \underline{\frac{d}{dr}(r^2\Omega)^2 \geq 0} \quad \text{as req}^{d}.$$

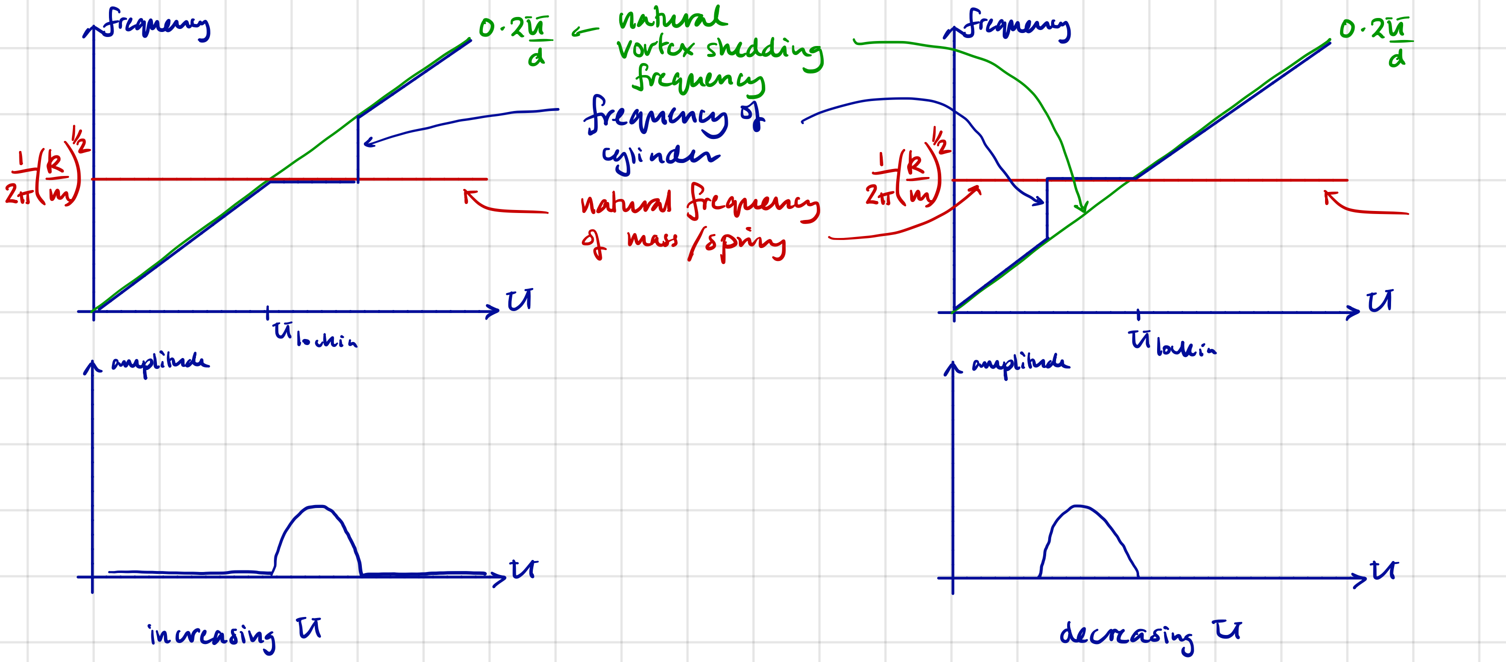
4. (a)



Natural frequency of mass-spring system $= \omega_n = \left(\frac{k}{m}\right)^{1/2}$ rad s^{-1} ; $f_n = \frac{1}{2\pi}\left(\frac{k}{m}\right)^{1/2}$

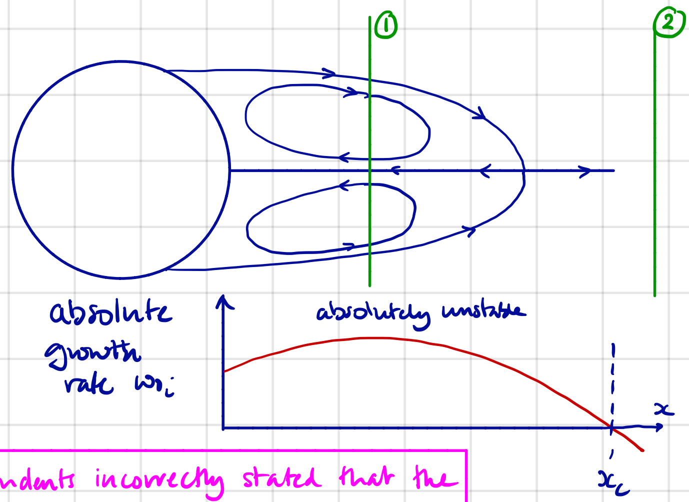
Strouhal number (based on f in Hz) $= \frac{fd}{U} = 0{\cdot}2$ for vortex shedding over a wide range of Re.

The rod will shed vortices at $f \simeq \frac{0{\cdot}2U}{d}$ Hz. When the frequency of vortex shedding matches the natural frequency of the mass/spring system, the mass will start to oscillate with large amplitude. The vortex shedding frequency will lock on to the mass/spring natural frequency and the amplitude will increase as U increases. Eventually, the natural vortex shedding frequency will exceed the mass/spring natural frequency so much that they will lock out. The amplitude will drop:



When the speed decreases, the same behaviour is observed, but for $U < U_{lock\,in}$, rather than for $U > U_{lock\,in}$

(b) There is a recirculation zone behind the rod: This flow is absolutely unstable in the recirculation zone and slightly beyond the recirculation zone. The large region of absolute instability causes the flow to oscillate at a well-defined frequency (given by Strouhal $\simeq 0{\cdot}2 = fd/U$).



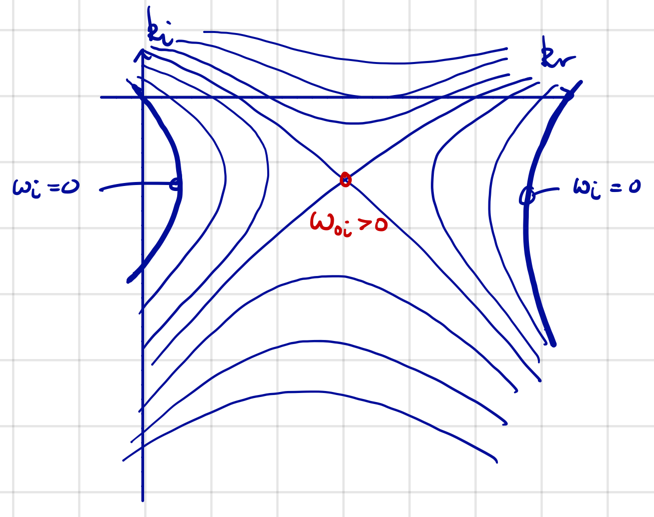
Some students incorrectly stated that the flow is absolutely unstable only when the shedding frequency has locked on to the oscillating frequency.

[cont...

This oscillator is insensitive to small amplitude forcing of the cylinder but can be overwhelmed by large amplitude forcing. This is what happens when the vortex shedding frequency locks into the mass/spring oscillation.

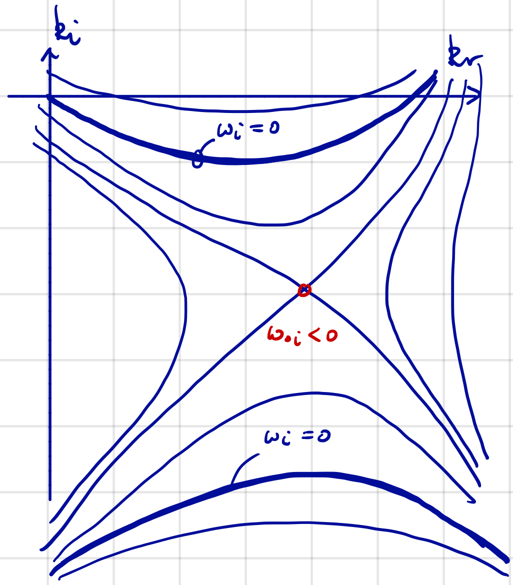
Contours of complex angular frequency are:

i) absolutely unstable region.



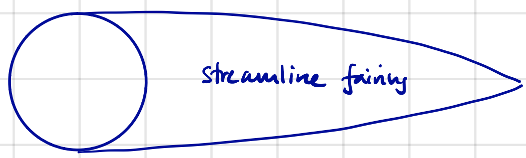
Saddle point has $\omega_i > 0$

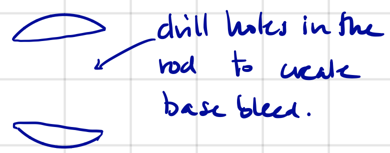
ii) convectively unstable region



Saddle point has $\omega_i < 0$.

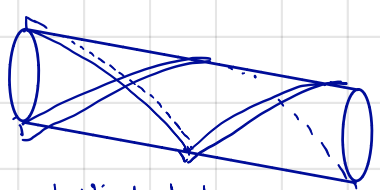
(c) With reference to part (b), the oscillations can be reduced by reducing the strength or size of the absolutely unstable region behind the rod. This can be done by streamlining the rod, to remove the wake, or by allowing air to pass through the rod, which is known as "base bleed":





With reference to part (a), the oscillations can be reduced by decorrelating vortex shedding along the length of the rod, or changing the natural frequency by varying d along the length of the rod, e.g. with strakes or a varying d:

Many students forgot about adding devices to suppress vortex shedding.



helical strakes

varying d.

Apart from the two comments above, this question was answered well by almost all students. Some wrote much more than was asked for in the question, which will have cost them time.

Matthew Juniper
Jan 2019

ENGINEERING TRIPOS PART IIB 2019
DETAILED COMMENTS, MODULE 4A10

Question 1
This question really required some thought and was attempted by two thirds of the class. The students generally gave good physical interpretations for the growth rate of the instability given and reasoning for the axisymmetric and non-axisymmetric modes predicted. Evidently, based on their clear descriptions, the vast majority had a good grasp of linear stability analysis.

Question 2
This was predominantly a discussion question, with a straightforward calculation in (a), repetition of the notes in (b,c), and tests of conceptual understanding in (d,e).

Most students answered (a) well, although a common and elementary mistake was to write that the moment of inertia is proportional to d/2 rather than d^2/4.

Few students answered (b) well, despite it being repetition of the notes. Almost all students, however, answered (c) well, which is the more important question.

Answers to (d,e) were mixed. Around 1/3 of the students showed that they understood how to find the phase information from Y_0 and theta_0. Around 1/2 described how they would calculate the work done over a cycle. Almost all students saw that (e) would lead to a large matrix and around 1/4 correctly identified that the infinite limit becomes like a continuous system with no pinned boundaries or wave reflections.

Question 3
Attempted by all students, both parts of this question were tackled generally very well. The ‘show the following…’ parts were done well; by contrast, the ‘and hence…’ parts were either missed or not attempted by numerous candidates.

Question 4
This question was well answered by most students. A common mistake was to write that the flow is absolutely unstable only when the shedding frequency has locked on to the vibration frequency. Many students forgot that devices can be added to a shape to suppress vortex shedding.