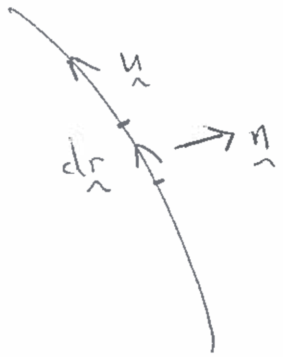
4A12 2019 Crib

1 (a)



$$\begin{cases} |\underset{\sim}{u}| = |\nabla\psi| \\ \nabla\psi \text{ is } /\!/^{\text{el}} \text{ to } n \end{cases}$$

So $|\nabla\psi\cdot\underset{\sim}{n}| = |\underset{\sim}{u}|$

Thus $\underline{\underline{\underset{\sim}{u}\cdot d\underset{\sim}{r} = |\underset{\sim}{u}||d\underset{\sim}{r}| = |\nabla\psi\cdot\underset{\sim}{n}||d\underset{\sim}{r}|}}$

(b) $\underset{\sim}{u}\cdot\nabla T = \alpha\nabla^2 T$

Small α $\Rightarrow$ $\underset{\sim}{u}\cdot\nabla T \simeq 0$

$\Rightarrow$ $T \approx$ constant on streamline

$\Rightarrow$ $T = T(\psi)$

$\nabla\cdot(T\underset{\sim}{u}) = \underset{\sim}{u}\cdot\nabla T = \alpha\nabla^2 T = \alpha\nabla\cdot(\nabla T)$

$\Rightarrow$ $\nabla\cdot(T\underset{\sim}{u}) = \nabla\cdot(\alpha\nabla T)$

But $\nabla T = \dfrac{dT}{d\psi}\nabla\psi$ $\Rightarrow$ $\underline{\underline{\nabla\cdot(T\underset{\sim}{u}) = \nabla\cdot\left(\alpha\frac{dT}{d\psi}\nabla\psi\right)}}$

(c)

$$\oint_C T\underset{\sim}{u}\cdot d\underset{\sim}{s} = \oint_C \alpha\frac{dT}{d\psi}\nabla\psi\cdot d\underset{\sim}{s}$$

(using Gauss)

But $\underset{\sim}{u}\cdot d\underset{\sim}{s} = 0$ and $\dfrac{dT}{d\psi}$ = constant on C

So $\underline{\underline{\alpha\dfrac{dT}{d\psi}\oint_C \nabla\psi\cdot\underset{\sim}{n}\,dr = 0}}$ $\quad (d\underset{\sim}{s} = dr\,\underset{\sim}{n})$

But $|\nabla\psi \cdot \underset{\sim}{n}|\, dr = \underset{\sim}{u} \cdot d\underset{\sim}{r}$ (from (a))

So, $$\alpha \frac{dT}{d\psi} \oint_C \underset{\sim}{u} \cdot d\underset{\sim}{r} = 0$$

Since α and $\oint_C \underset{\sim}{u} \cdot d\underset{\sim}{r}$ are both non-zero, $\underline{\underline{\frac{dT}{d\psi} = 0}}$

This means T does not vary across the streamlines.

(d) Theorem states that, for a <u>steady</u>, <u>2D</u> flow at <u>large Re</u>, $\underset{\sim}{\omega}$ is uniform in a region of <u>closed streamlines</u>. It does not apply in boundary layers. The proof rests on the steady 2D equation

$$\underset{\sim}{u} \cdot \nabla \omega = \nu \nabla^2 \omega$$

Repeat analysis of (c) but $T \leftrightarrow \omega$, $\alpha \leftrightarrow \nu$.

<u>Interpretation:</u> $\underset{\sim}{u} \cdot \nabla\omega \approx 0 \Rightarrow \omega = \omega(\psi)$

(fluid blobs retain a constant value of ω as they circulate around a closed streamline)

A slow cross-stream diffusion of ω then slowly erradicates any gradients in ω, leaving ω uniform

(e) Turbulent diffusion of T and ω replaces molecular diffusion, i.e.

$$\bar{\underset{\sim}{u}} \cdot \nabla \bar{\underset{\sim}{\omega}} = \underset{\substack{\uparrow \\ \text{eddy viscosity}}}{\nu_t} \nabla^2 \bar{\underset{\sim}{\omega}}$$

(2) (a)

Law 1: The vortex lines move with the fluid, like dye lines

Law 2: The flux of vorticity along a vortex tube is the same for all cross-sections of the tube and independant of time,

These laws apply only to an inviscid fluid.

(b)

$$\frac{D\underset{\sim}{B}}{Dt} = (\underset{\sim}{B}\cdot\nabla)\underset{\sim}{u}, \quad \nabla\cdot\underset{\sim}{B} = 0$$

c.f.

$$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u}, \quad \nabla\cdot\underset{\sim}{\omega} = 0 \quad \text{(Identical)}$$

Thus $\underset{\sim}{B}$ and $\underset{\sim}{\omega}$ must behave in identical ways for a given velocity field $\underset{\sim}{u}$. Thus.

1. The $\underset{\sim}{B}$-lines will move with the fluid, like dye lines
2. The magnetic flux in a flux tube will be the same at all cross-sections and independant of time.

(c)

$$\frac{D}{Dt}d\underset{\sim}{r} = (d\underset{\sim}{r}\cdot\nabla)\underset{\sim}{u}$$

Since $d\underset{\sim}{r}$ and $\underset{\sim}{B}$ obey identical equations, $\underset{\sim}{B}$ must evolve in the same way as $d\underset{\sim}{r}$ for a given velocity field, $\underset{\sim}{u}$. But $d\underset{\sim}{r}$ is part of a dye line, so $\underset{\sim}{B}$ moves with the fluid, like dye.

(d)

$$H = \int_{C_1} \underset{\sim}{A}\cdot\underset{\sim}{B}\,dV + \int_{C_2} \underset{\sim}{A}\cdot\underset{\sim}{B}\,dV$$

But $\underset{\sim}{B}\,dV = \Phi\,d\underset{\sim}{r}$

$$\Rightarrow \quad H = \oint_{C_1} \underset{\sim}{A}\cdot(\Phi_1 d\underset{\sim}{r}) + \oint_{C_2} \underset{\sim}{A}\cdot(\Phi_2 d\underset{\sim}{r})$$

But Φ_1 and Φ_2 are constant along C_1 and C_2, so

$$H = \Phi_1 \oint_{C_1} \underset{\sim}{A} \cdot d\underset{\sim}{r} + \Phi_2 \oint_{C_2} \underset{\sim}{A} \cdot d\underset{\sim}{r}$$

For right-handed linkage, Stokes gives

$$\oint_{C_1} \underset{\sim}{A} \cdot d\underset{\sim}{r} = \Phi_2, \quad \oint_{C_2} \underset{\sim}{A} \cdot d\underset{\sim}{r} = \Phi_1$$

So $$\underline{\underline{H = 2\Phi_1\Phi_2}}$$

(e) The analogue of Helmholtz's laws tell us that Φ_1 and Φ_2 are invariant

Also, the two flux tubes must remain linked for all time.

Thus $H = 2\Phi_1\Phi_2 = $ constant.

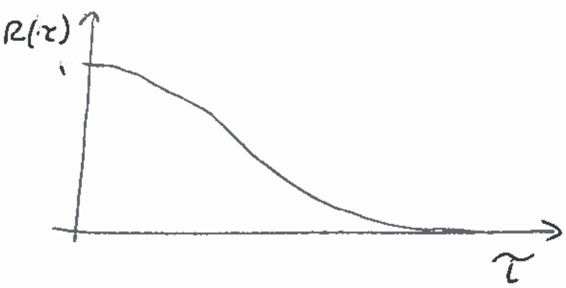
Q3

(a) Energy cascade refers to the idea that the turbulent kinetic energy flows across eddy range without any dissipation until the smallest (Kolmogorov) scales, where it is lost to heat. Hence, at the largest eddies, energy is removed from the mean flow through the action of the turbulent velocity fluctuations, it is then transferred to eddies of progressively smaller sizes, and the smallest eddy size is then determined so that the velocity gradients there are such that to dissipate the right amount of energy, as determined by the energy fed from the large scales. The usual model for the turbulent kinetic energy dissipation per unit mass, ε, is $\varepsilon = k^{3/2}/L$ (given in the Data Card). This is based on the idea that an amount of kinetic energy per unit mass k, is to be dissipated (or, alternatively, fed down to the small scales) at a rate equal to the large-eddy turnover time $L/k^{1/2}$, which gives $\varepsilon = k / (L/k^{1/2})$. Any constants in the above expression are of order unity and do not concern us in this order-of-magnitude estimate.

(b) If the kinematic viscosity changes, the dissipation does not change because it is solely determined by L and k, which are large-scale quantities that are not affected by viscosity. The Taylor lengthscale λ is defined from the relation $\varepsilon = 15\nu k/\lambda^2$, hence halving the viscosity for the same k and ε means λ is reduced by a factor of $\sqrt{2}$. Similarly, for the Kolmogorov lengthscale $\eta_K = (\nu^3/\varepsilon)^{1/4}$ (from the Data Card), reducing n by a factor of 2 means η_K is reduced by a factor of $2^{3/4}$.

(c) The integral timescale is defined as the integral under the autocorrelation curve of the velocity signal, as shown below:

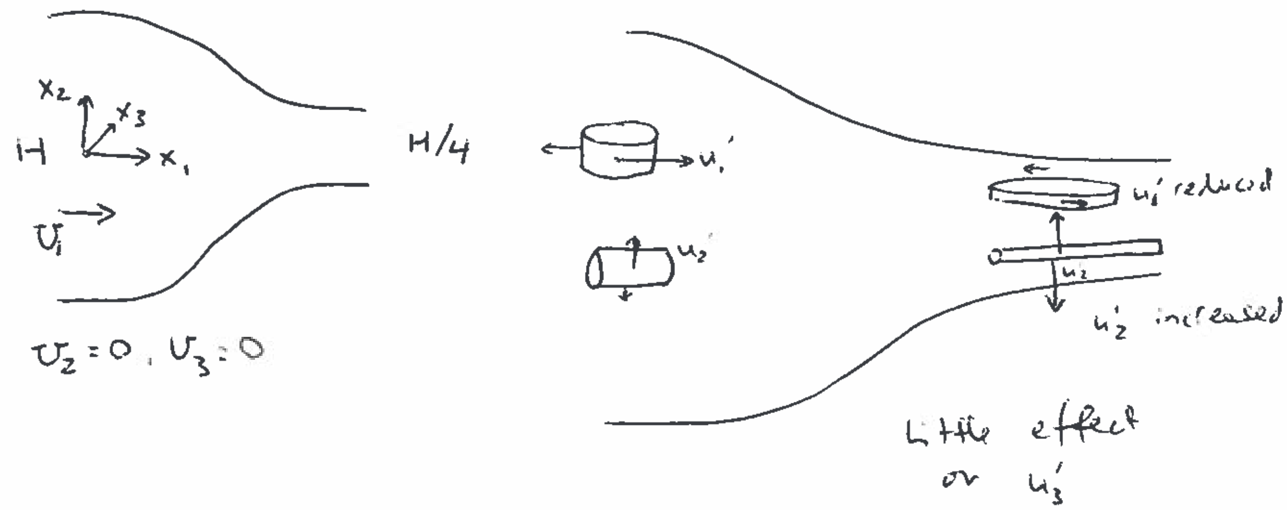
$$R(\tau) = \frac{\overline{u'(t)\,u'(t+\tau)}}{\overline{u'^2}}$$



$$T' = \int_0^\infty R(\tau)\,d\tau$$

The hot wire must be able to resolve the velocity fluctuation across the smallest eddy as it is being convected past the probe by the mean velocity U, hence the fastest frequency must be U/η_K [redacted]. The integral timescale T' is related to the eddy turnover time $T = k/\varepsilon$ (or $L/k^{1/2}$) by $T' = L/U$.

Q4



(a) For this flow where there is no shear and no body force, the production terms of the individual Reynolds stresses are: $P_1 = -\overline{u_1' u_1'}\frac{dU1}{dx1}$, $P_2 = -\overline{u_2' u_2'}\frac{dU2}{dx2}$, and P_3=0. Hence, since U_1 increases along the contraction, the streamwise normal stress decreases (the production term is negative), while the cross-stream normal stress increases (the production term is positive). The production term of the component normal to the plane is zero, hence to first approximation, there is no effect on the third component.

(b) Conservation of angular momentum implies a "barrel"-shaped eddy gets distorted by the mean flow, but maintains its angular momentum. Hence, the angular velocity of the eddy changes when its diameter changes, as shown in the diagram above. This then allows estimates to be made for the streamwise and cross-stream turbulence components as shown in the diagram.

Comments on questions:

Question 1: The Prandtl-Batchelor theorem
Popular question done by all candidates. Generally done very well, though many candidates had trouble with the first part.

Question 2: Helmholtz's laws applied to magnetic fields
This was a difficult and unfamiliar question involving magnetic fields. Nevertheless it was a popular question done by all but 2 candidates. Surprisingly well done.

Question 3: Energy cascade
Generally answered very well, with few students though including vortex stretching in their discussion of the energy cascade and energy transfer

Question 4: Turbulence in a contraction
Very few students answered this question. The second part was done very well by most students; everybody seemed to appreciate vortex stretching ideas.