Worked solutions for 4B23 Optical Fibre Communication 2018/19 paper version 1.3

Assessors comments:
Question 1: This question was answered by more than half of the candidates. It dealt with a few-mode multicore fibre, with the first part of the question covering modal analysis of a few mode fibre, before considering coupling between adjacent cores and culminating with the design of a multicore multimode fibre. Generally this question was answered very well, albeit in part b) some candidates failed to provide both limits over which the waveguide would supported exactly four modes. The final section which required an element of design / estimation was answered well by a handful of candidates.

Question 2: This question was answered by less than half of the candidates, dealing with noise in optically amplified systems. While the first two parts were answered well, part c) which required students to analyse signal to noise when transceiver impairments are included, was handled poorly by many students that attempted the question – failing for example to use a result that was successfully used by all students in the previous part of the question. Part d) was also handled poorly with no students, considering for example the differential noise added over a differential distance to determine how the SNR varied with distance (or even a simpler approach of calculating the noise added for 1 km)

Question 3: This question was answered by all candidates. The first half of the question relating to digital dispersion compensation was answered well by all candidates (albeit errors did occur such as using symbol rate as opposed to sample rate to calculate the total power or forgetting to calculate the power for the two polarisations). The answer to part c) which required the signal to noise to be calculated was answered with varying success, a few candidates correctly used the various formulae in order to calculate the optimal launch power, others guessed at values orders of magnitude higher than might be expected, while one candidate used the DSP power to perform digital dispersion compensation as the optimal launch power! The final part of this question was in general not answered well in part because students had failed to understand how to answer part c) correctly.

WORKED SOLUTIONS

1. (a) In a step index optical fibre, the dimensionless parameter, the normalised wavenumber V determines the number of modes that the fibre supports. For a given value of V by considering the zeros of the first order Bessel function $J_n(x)$ the exact number of modes can be calculated. For the linearly polarised modes LP_{mn} the approximate number of modes scales as $N \approx (V/\pi)^2$ whereas for true modes the number of modes scales as $N \approx 0.5 \times V^2$, however in both cases the number of modes scales quadratically with V.

   (b) For the fibre specified the normalised wavenumber V at 1550 nm is given by

$$V = \frac{\pi d}{\lambda}\sqrt{n_{co}^2 - n_{cl}^2} = \pi \times \frac{10}{1.55}\sqrt{1.4545^2 - 1.44^2} = 4.224$$

There are only two zeros of the Bessel function which are less than this corresponding to $J_0(2.405) = 0$ and $J_1(3.832) = 0$. LP_{01} will propagate, and since $V > 2.405$ then LP_{11} will propagate. For $J_1(3.832) = 0$, this corresponds to $m = 2$ but also $m = 0$ and therefore LP_{21} and LP_{02} will propagate. Hence the four modes which propagate are LP_{01}, LP_{11}, LP_{02} and LP_{21}

The next zero of the Bessel functions occurs at $J_2(5.136) = 0$ corresponding to $m = 3$ and the mode LP_{31}. For the fibre to support exactly four modes we therefore need $3.832 < V < 5.136$, and noting $\lambda = \pi \times \frac{10}{V}\sqrt{1.4545^2 - 1.44^2} = 6.436/V$ so

$$1.253\ \mu m < \lambda < 1.680\ \mu m$$

c)

i) Coupled mode approach between two modes of amplitude A_1 and A_2 gives

$\frac{dA_1}{dz} = -j\kappa A_2$ & $\frac{dA_2}{dz} = -j\kappa A_1$ so $\frac{d^2A_1}{dz^2} = -\kappa^2 A_1$ i.e. $A_1 = a\cos\kappa z + b\sin\kappa z$

Therefore the proportion of the field that couples from one core to another is given by $\sin\kappa z$ so the proportion of power is $\sin^2\kappa z$.

For $\kappa z \ll 1$ we know $\sin^2\kappa z \approx (\kappa z)^2$ and therefore the proportion of power coupled is $(\kappa z)^2$. In a decibel scale this is $10\log_{10}(\kappa z)^2 = 20\log_{10}\kappa z$ as required.

ii) $\kappa z = \frac{z}{a^2 n_{co} k_0} \times V^{(2-\rho^2)}$. Given at $\lambda = 1550$ nm gives $V = 4.224$ then for $z = $1km

$$\begin{aligned}20\log_{10}\kappa z &= 20\log_{10}\frac{z}{a^2 n_{co} k_0} + 40\log_{10} V - 20\rho^2\log_{10} V \\ &= 20\log_{10}\frac{10^3}{25\times 10^{-12}\times 1.44\times\frac{2\pi}{1.55}\times 10^6} + 40\log_{10} 4.224 \\ &\quad - 40\,\rho\log_{10} 4.224 = 136.7 + 25 - 12.5\rho^2 = 161.7 - 12.5\rho^2\end{aligned}$$

Hence for $-60 = 161.7 - 12.5\rho^2$ gives ρ^2 =17.76 and hence minimum value for ρ is 5.

iii) Consider a 7-core fibre in which the central core receives crosstalk from 6 adjacent cores (corresponding to $10\log_{10} 6$=7.8 dB in crosstalk). With $\rho = 5$ the cross talk is well below the -60 dB required. If we assume that cores are arranged in a hexagon then 19 cores (each separated by 25 µm) fits inside a diameter of 100µm so an estimate would be 19 cores (each supporting 4 modes so 76 fold increase in capacity!).

N.B. Using the minimum number for ρ which would be 4.28 (based on crosstalk of $-67.8 = -60 - 10\log_{10} 6$) gives a core separation of 21.4 µm. Solving this circle packing problem would in principle allow for 26 cores but the assumption of three cores forming equilateral triangle, gives 19 as the maximum number of cores.

2. a)

   i) Pump laser is used to excite photons to an excited state, typically the wavelength of the pump laser is either 1480 nm or 980 nm.

   ii) The gain of the amplifier is the ratio of photon flux at the output of the amplifier to the photon flux at the input of the amplifier

   iii) Amplified spontaneous emission (ASE) is the noise generated by the optical amplifier

   iv) The noise figure is the ratio of the input SNR to the output SNR given in decibels.

b) Input signal to noise ratio (SNR) is

$$SNR_{in} = \frac{S_{in}}{N_{in}}$$

Suppose we now amplify the signal with a gain G. In this case the output signal will be $S_{out} = GS_{in}$ and the output noise will be $N_{out} = GN_{in} + N_a$ where N_a is the noise added by the amplifier. Hence the output SNR is given by

$$SNR_{out} = \frac{P_{out}}{N_{out}} = \frac{GS_{in}}{GN_{in} + N_a}$$

we define the noise figure $NF = 10\log_{10} F$ where F is the noise factor of the amplifier, which is defined to be the ratio of the ***input*** SNR to the ***output*** SNR, such that

$$F = \frac{SNR_{in}}{SNR_{out}} = \frac{S_{in}}{N_{in}}\frac{GN_{in} + N_a}{GS_{in}} = 1 + \frac{N_a}{GN_{in}}$$

The minimum input noise per polarisation is due to the shot noise such that over a bandwidth B centred about a frequency ν then $N_{in} = h\nu B$ and the minimum noise power $N_a = h\nu(G-1)B$ gives

$$F = 1 + \frac{(G-1)}{G}$$

Therefore for a high gain amplifier with $G \gg 1$ the minimum noise factor is 2 and hence the minimum noise figure is 3 dB.

c)

Note: from $SNR_{out} = \frac{GS_{in}}{GN_{in}+N_a}$ it follows that $\frac{1}{SNR_{out}} = \frac{1}{SNR_{in}} + \frac{N_a}{GS_{in}}$

i) noise from the high gain amplifier is $2h\nu \times 99 \times 31.5 \times 10^9 = 0.8\mu$W= -31.0 dBm therefore SNR of amplifier is 29.0 dB. Given the original SNR was 25 dB then the resulting SNR is

$$-10\log_{10}(10^{-2.9} + 10^{-2.5}) = 23.5\ dB$$

ii) noise from the low gain amplifier is $4h\nu \times 9 \times 31.5 \times 10^9 = 145$ nW= -38.4 dBm. At the output of the first amplifier the signal power is -12 dBm therefore the SNR of first amplifier is 26.4 dB. At the output of the second amplifier the signal power is -2 dBm so the SNR of the second amplifier is 36.4 dB. Given the original SNR was 25 dB then the resulting SNR is

$$-10\log_{10}(10^{-2.64} + 10^{-3.64} + 10^{-2.5}) = 22.5\ dB$$

iii) noise from the low gain amplifier is $4h\nu \times 9 \times 31.5 \times 10^9 = 145$ nW= -38.4 dBm therefore and the optical power after the low gain amplifier after 40 km is 0 dBm SNR of amplifier is 38.4 dB. After the second amplifier the noise is -38.4 dBm again but the signal is now +2 dBm so the SNR is 40.4 dB Given the original SNR was 25 dB then the resulting SNR is

$$-10\log_{10}(10^{-3.84} + 10^{-4.04} + 10^{-2.5}) = 24.7\ dB$$

N.B. The question states nonlinearity can be neglected at the power levels transmitted. While strictly even with a low nonlinearity fibre ($A_{eff} = 150\ \mu m^2$ and D=21 ps/nm/km), there will be some nonlinearities at 0 dBm and +2 dBm, due to transceiver noise the result of including nonlinearities is similar on the overall SNR i.e. SNR of 24.6 dB justifying this approximation.

d) Using Shannon's theorem to determine the SNR needed to transmit 100 Tbit/s in 5 THz gives

$$100 \times 10^{12} = 2 \times 5 \times 10^{12} \times \log_2(1 + SNR)$$

Therefore $SNR = 2^{10} - 1 = 1023$

To estimate the distance at which the SNR is 1023, we first calculate the power

Assuming the same initial power spectral density as for part (c) i.e. in 31.5GHz a power of -2dBm=0.63 mW so 20 mW/THz so in 5THz the power is 100 mW.

To determine the noise, we consider the differential noise $\delta\sigma_{ASE}^2$ added over a differential distance δz and use a Taylor series for the exponential such that

$$\delta\sigma_{ASE}^2 = 2h\nu\left(e^{\alpha\delta z} - 1\right)B \approx 2h\nu\alpha B\delta z$$

Therefore the total noise over a length L is

$$\sigma_{ASE}^2 = 2h\nu\alpha BL$$

For α_{dB}=0.15 dB/km gives $\alpha = 0.23 \times 0.15 = 0.0345$ km^{-1}

Therefore if L is in km we have

$$\sigma_{ASE}^2 = 2 \times 1.3 \times 10^{-19} \times 0.0345 \times 5 \times 10^{12} \times L = 4.485 \times 10^{-8}L$$

Therefore

$$1023 = \frac{100 \times 10^{-3}}{4.485 \times 10^{-8}L}$$

Hence

$$L = \frac{100 \times 10^{-3}}{4.485 \times 10^{-8} \times 1023} = 2180 \text{ km}$$

3.

a) Rectangular Nyquist spectrum means that a 100 GBd signal will occupy 100 GHz of optical spectrum. At 1550 nm, 100 GHz occupies 0.8 nm so the chromatic dispersion from 1000 km is

$$21 \times 0.8 \times 1000 = 16.8 \text{ ns}$$

The sampling rate is $100 \times \frac{8}{7} = 114.3$ GSa/s therefore the number of samples corresponding to 16.8 ns is

$N_{CD} = 16.8 \times 10^{-9} \times 100 \times 10^{9} \times \frac{8}{7} = 1920$ samples

b) Using the overlap and save algorithm with an N point FFT the number of complex multiplies per sample N_{cm} is

$$N_{cm} = \frac{N \log_2(N) + N}{N - N_{CD} + 1}$$

Given $N_{CD} = 1920$ we expect the minimum value of N to be 4096 which gives

$$N_{cm} = \frac{4096 \times 12 + 4096}{4096 - 1920 + 1} = 24.5$$

Given there are no technological limitations regarding the FFT size let us consider $N = 8192$ which gives

$$N_{cm} = \frac{8192 \times 13 + 8192}{8192 - 1920 + 1} = 18.3$$

Increasing to $N = 16384 = 2^{14}$ gives

$$N_{cm} = \frac{2^{14} \times 14 + 2^{14}}{2^{14} - 1920 + 1} = 16.99$$

Increasing to $N = 2^{15}$ gives

$$N_{cm} = \frac{2^{15} \times 15 + 2^{15}}{2^{15} - 1920 + 1} = 17.00$$

Hence the optimum value of N is 16384. The power consumption per polarisation is

$P = 16.99 \times 10^{-12} \times 114.3 \times 10^{9} = 1.94\ W$ and hence for two polarisations the total power consumption is 3.9 W.

c) For span of 100 km $L_{eff} \approx \frac{1}{\alpha} = \frac{1}{0.23 \times 0.16} = 27$ km

The nonlinear coefficient of the fibre is $\gamma = n_2 k_0 / A_{eff}$ where

$n_2 = 2.6 \times 10^{-20}$ m²/W and hence

$$\gamma = 2.6 \times 10^{-20} \times \frac{2\pi}{1550 \times 10^{-9}} \times \frac{1}{150 \times 10^{-12}} = 0.7\ km^{-1}W^{-1}$$

If c is in nm/ps then $c = 3 \times 10^5$ nm/ps
$\beta_2 = -\frac{\lambda^2}{2\pi c} D = -21 \times \frac{1550^2}{2\pi \times 3 \times 10^5} = -26.8$ ps²/km

Hence $C_{NLI} = \frac{8\gamma^2 L_{eff}^2 \alpha}{27\pi|\beta_2|} \ln\left(\frac{|\beta_2|}{\alpha}\pi^2 B^2\right)$ with $B = 0.1\ THz$ so

$$C_{NLI} = 8 \times 0.7^2 \times \frac{27}{27\pi 27} \ln(27 \times 27 \times \pi^2 \times 0.1^2) = 0.2\ pJ^{-2}$$

Likewise

$$N_{ASE} = 4 \times 0.8 \times 1.6 \times 10^{-19} \times (40 - 1) = 2 \times 10^{-5}\ pJ$$

Therefore optimum transmitted PSD is $\sqrt[3]{\frac{(2\times10^{-5})}{2\times0.2}} = 0.0368\ pJ$

For the 100 GBd PDM-64QAM signal the optimum launch power is 3.68 mW (since 1 pJ=1 mW/GHz).

The maximum SNR at this launch power is $\frac{0.0368}{3\times10^{-5}} = 1228$ (since NL contribution is 50% of ASE at optimum), and hence after 10 spans we have an SNR of 122.8 corresponding to 20.9 dB.

d) Recognise that part (c) gives the desired SNR so could reduce the number of amplifiers. The nonlinear noise term is unchanged but with 9 inline amplifiers the span length becomes 111.1 km requiring gain of 60 required giving $N_{ASE} = 4 \times 0.8 \times 1.6 \times 10^{-19} \times (60 - 1) = 3 \times 10^{-5}\ pJ$

Hence optimum transmitted PSD is $\sqrt[3]{\frac{(3\times10^{-5})}{2\times0.2}} = 0.0422\ pJ$

Therefore SNR after 9 spans becomes $\frac{0.0422}{9\times\frac{3}{2}\times3\times10^{-5}} = 104 = 20.2\ dB$

Deduce that 9 is the minimum number of in-line amplifiers.

To check examine 8 in-line amplifiers with span length of 125 km requiring gain of 100 giving $N_{ASE} = 4 \times 0.8 \times 1.6 \times 10^{-19} \times (100 - 1) = 5 \times 10^{-5}\ pJ$

Hence optimum transmitted PSD is $\sqrt[3]{\frac{(5\times10^{-5})}{2\times0.2}} = 0.05\ pJ$

Therefore SNR after 8 spans becomes $\frac{0.05}{8\times\frac{3}{2}\times5\times10^{-5}} = 83.3 = 19.2\ dB$