Engineering Tripos, Part IA, 2015

Paper 1 Mechanical Engineering

Solutions

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1 (a) The internal energy of the separate blocks must be equal to the combined internal energy of the blocks when together. Therefore, if c_m is the specific heat capacity of the metal,

$$T_F = \frac{c_m m_A T_A + c_m m_B T_B}{c_m (m_A + m_B)} = \frac{m_A T_A + m_B T_B}{(m_A + m_B)}$$ [3]

(b) The maximum work will be delivered if a reversible heat engine is employed and the final temperatures of the blocks are equal.
For a reversible heat engine the work δW produced from δQ units of heat delivered from the hotter block is given by,

$$\frac{dW}{dQ} = \frac{T_H - T_C}{T_H}$$

Also, $dQ = -m_A c_m dT_H$, $dQ - dW = m_B c_m dT_C$, and so $dW = -m_A c_m dT_H - m_B c_m dT_C$.

Hence,

$$\frac{m_A dT_H + m_B dT_C}{m_A dT_H} = 1 + \frac{m_B dT_C}{m_A dT_H} = \frac{T_H - T_C}{T_H} = 1 - \frac{T_C}{T_H}, \quad \text{or} \quad m_A \frac{dT_H}{T_H} + m_B \frac{dT_C}{T_C} = 0$$

Integrating,

$$m_A \ln(T_H) + m_B \ln(T_C) = const = m_A \ln(T_A) + m_B \ln(T_B) \quad \text{(from the initial condition)}$$

The final temperature is when $T_F = T_H = T_C$. Hence,

$$\ln(T_F) = \frac{m_A}{(m_A + m_B)} \ln(T_A) + \frac{m_B}{(m_A + m_B)} \ln(T_B), \quad \text{or} \quad T_F = (T_A)^r (T_B)^{1-r}$$ [7]

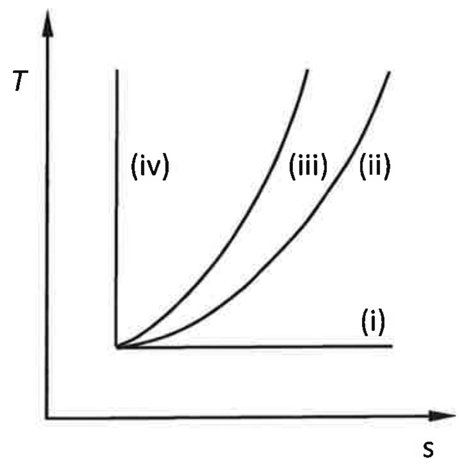
2 (a) For a reversible process, $dq_{rev} = Tds$ and $dw_{rev} = pdv$, so $dq = du + dw$ may be written $Tds = du + pdv$. From the definition $h = u + pv$ we have $dh = du + pdv + vdp$ and then, by substitution for $du + pdv$, we obtain $Tds = dh - vdp$. This, and the $Tds = du + pdv$ expression, also holds for irreversible processes as there is always, via heat and work transfers, a reversible route between states, and irreversible processes between states cannot result in a different relationship between properties, since the properties define the state. [3]

(b) (i) Constant enthalpy. For a perfect gas $h = c_pT$ + constant. Hence, a constant *h* line is also a constant *T* line which is a horizontal line on a *T-s* diagram.

(ii) Constant pressure. From the relationship above, $Tds = dh - vdp = dh + 0$.
For a perfect gas $dh = c_p dT$, so $\frac{dT}{ds} = \frac{T}{c_p}$. Hence, the trend is one of positive slope with the gradient increasing with temperature.

(iii) Constant volume. From the relationship above, $Tds = du + pdv = du + 0$.
For a perfect gas $du = c_v dT$, so $\frac{dT}{ds} = \frac{T}{c_v}$. Since $c_p > c_v$ always, the trend for this process is similar to that for a constant pressure process, but with an increased gradient.

(iv) Reversible and adiabatic. This is equivalent to isentropic, *i.e.*, a vertical line on a *T-s* diagram.



[7]

3 (a) From tables for argon, R = 208 J/kg K and c_v = 310 J/kg K.

Argon is a perfect gas and so the initial volume is given by,

$$V_1 = \frac{mRT_1}{p_1} = \frac{0.01\times 208\times 300}{10^5} = 0.00624 \text{ m}^3$$ [6]

(b) The piston area is $A = \frac{\pi D^2}{4} = \frac{\pi\times 0.1^2}{4} = 0.00785 \text{ m}^2$

The force exerted by the spring on the piston after compression over a distance L is $F = kL$.

If the gas pressure is p_2 then a force balance on the frictionless piston gives $p_2A = p_1A + F$.

Hence, the final pressure, volume and temperature are given by,

$$p_2 = p_1 + \frac{kL}{A} = 10^5 + \frac{200\times 0.5}{0.00785} = 1.127\times 10^5 \text{ N/m}^2$$

$$V_2 = V_1 + AL = 0.00624 + 0.00785\times 0.5 = 0.0102 \text{ m}^3$$

$$T_2 = \frac{p_2V_2}{mR} = \frac{1.127\times 10^5\times 0.0102}{0.01\times 208} = 552.7 \text{ K}$$

The work done by the gas is equal to the work done in compressing the spring plus the work done against the atmosphere at pressure p_1 :

$$W_{12} = \frac{kL^2}{2} + p_1AL = \frac{200\times 0.5^2}{2} + 10^5\times 0.00785\times 0.5 = 25.0 + 392.5 = 417.5 \text{ J}$$

The increase in internal energy of the gas and the heat transerred to the gas are given by,

$$U_2 - U_1 = mc_v(T_2 - T_1) = 0.01\times 310\times(552.7 - 300) = 783.3 \text{ J}$$

$$Q_{12} = W_{12} + (U_2 - U_1) = 417.5 + 783.3 = 1200.8 \text{ J}$$ [12]

(c) The motion of the piston of mass M will be governed by Newton's 2nd law. If the small distance moved from the initial position is Δx then the pressure in the cylinder will change to a pressure $p_1+\Delta p$ and the net force on the piston will be $(p_1+\Delta p)A - p_1A - k\Delta x = A\Delta p - k\Delta x$. The motion is therefore described by the ODE,

$$M\frac{d^2(\Delta x)}{dt^2} = -k\Delta x + A\Delta p$$

The changes in the gas are isentropic and so pV^γ = constant. Taking logs and differentiating gives, for small changes, $\Delta p/p_1 \cong -\gamma\Delta V/V_1 = -\gamma A\Delta x/V_1$. Hence,

$$M\frac{d^2(\Delta x)}{dt^2} = -\left(k + \frac{\gamma p_1 A^2}{V_1}\right)\Delta x$$

By inspection, the effective spring constant is $(k + \gamma p_1 A^2/V_1)$. [12]

4 (a) $w = f_0(V_1, \quad U, \quad T_1, \quad c_p, \quad R)$

SI units : $\frac{\text{J}}{\text{kg}} \quad \frac{\text{m}}{\text{s}} \quad \frac{\text{m}}{\text{s}} \quad \text{K} \quad \frac{\text{J}}{\text{kgK}} \quad \frac{\text{J}}{\text{kgK}}$ $\quad \left(\text{J} \equiv \text{Nm} = \frac{\text{kg m}^2}{\text{s}^2} \right)$

The dimensions of the variables are :

$$w = f_0(V_1, \quad U, \quad T_1, \quad c_p, \quad R)$$

$$\frac{\text{L}^2}{\text{T}^2} \quad \frac{\text{L}}{\text{T}} \quad \frac{\text{L}}{\text{T}} \quad \theta \quad \frac{\text{L}^2}{\text{T}^2\theta} \quad \frac{\text{L}^2}{\text{T}^2\theta}$$

There are 6 variables and 3 dimensions (L, T, θ) so, according to Buckingham's rule, there are 6 – 3 = 3 <u>or more</u> non-dimensional groups required to specify the problem. [4]

(b) In performing the eliminations there are several possible options. The following produces the 'standard' set of groups but any combination is analytically correct.
First, note that V_1 and U, and c_p and R, have the same dimensions. Hence,

$$w = f_1\left(\frac{V_1}{U}, \quad U, \quad T_1, \quad c_p, \quad \frac{c_p}{R}\right)$$

$$\frac{\text{L}^2}{\text{T}^2} \quad - \quad \frac{\text{L}}{\text{T}} \quad \theta \quad \frac{\text{L}^2}{\text{T}^2\theta} \quad -$$

Eliminate θ using T_1 :

$$w = f_2\left(\frac{V_1}{U}, \quad U, \quad c_pT_1, \quad \frac{c_p}{R}\right)$$

$$\frac{\text{L}^2}{\text{T}^2} \quad - \quad \frac{\text{L}}{\text{T}} \quad \frac{\text{L}^2}{\text{T}^2} \quad -$$

Eliminate L using U (T goes at the same time) :

$$\frac{w}{U^2} = f_3\left(\frac{V_1}{U}, \frac{c_pT_1}{U^2}, \frac{c_p}{R}\right)$$

Four non-dimensional groups completely define the problem. The conventional set is :

$$\frac{w}{U^2} = f_4\left(\frac{V_1}{U}, \frac{U}{\sqrt{c_pT_1}}, \frac{c_p}{R}\right)$$ [6]

5 (a) The anti-clockwise torque about the pivot, exerted by the water on the dam, is given by (w = width = 0.5 m),

$$T_A = \int_1^3 \rho g w x(x-1)dx = \rho g w\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3 = \frac{14}{3}\rho g w$$

The clockwise torque about the pivot due to the mass M is given by $T_C = 5Mg$.
For the dam to open when the water level is 2 m,

$$5Mg = \frac{14}{3}\rho g w \quad \rightarrow \quad M = \frac{14}{15}\rho w = \frac{14 \times 1000 \times 0.5}{15} = 467 \text{ kg}$$ [5]

(b) The hole and valve should be placed at the lowest part of the dam so that the pressure difference driving the flow through the hole is a maximum.
The minimum flow area will correspond to a frictionless flow through the hole and valve for which Bernoulli's equation will apply. Applying Bernoulli's equation from far upstream on the bed where the pressure is ($p_{at} + 2\rho g$), the velocity is zero and the height (above the bed) is zero, to the exit plane of the hole where the pressure is p_{at} , the velocity is V and the height is zero, gives,

$$(p_{at} + 2\rho g) + 0 + 0 = p_{at} + \frac{\rho V^2}{2} + 0 \quad \rightarrow \quad V = \sqrt{4g}$$

The flowrate is given by $AV = 0.001 \text{ m}^3\text{s}^{-1}$ where A is the flow area at the valve. Hence,

$$A = \frac{0.001}{\sqrt{4g}} = 1.6 \times 10^{-4} \text{m}^2$$ [5]

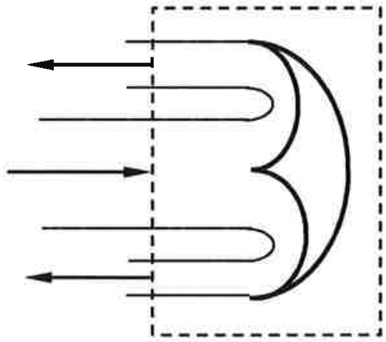
6. (a) With the deflector stationary and assuming the flow steady, incompressible and inviscid, Bernoulli's equation becomes,

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

Because $p_1 = p_2 = p_{at}$ it follows that $V_2 = V_1$. [4]

(b) Using steady-flow mass continuity, $\rho A_1 V_1 = \rho A_2 V_2$. Hence, $A_2 = A_1$. [5]

(c) Consider a control volume as sketched and use the Steady Flow Momentum Equation.



Force F = momentum flowrate (in – out)
$= \rho Q\,[V_1 - (-V_2)]$
$= 2\rho Q V_1$

where $Q = A_1 V_1$ is the volumetric flowrate. [7]

(d) If the deflector is moving to the right with velocity U then we move into a frame of reference moving with the deflector so that the flow is steady in the control volume sketched above. In this frame of reference, the velocity at entry to this control volume is $(V_1 - U)$. The area A_1 is unchanged so the volumetric flowrate into the control volume is $A_1(V_1 - U)$. Using the same analysis as for part (c), the force exerted by the water on the deflector is $F = 2\rho A_1 (V_1 - U)^2$. [7]

(e) The power extracted from the jet is the product of the force and the deflector speed,

$$P = FU = 2\rho A_1 (V_1 - U)^2 U$$

At maximum or minimum power,

$$\frac{dP}{dU} = 2\rho A_1[-2(V_1 - U)U + (V_1 - U)^2] = 0$$

$$\rightarrow 2\rho A_1 (V_1 - U)(V_1 - 3U) = 0$$

By inspection, maximum power is obtained when $U = V_1/3$. (The minimum is obtained when $U = V_1$). The assumptions are that the flow is incompressible and inviscid and that there is no loss of mechanical energy in the control volume containing the deflector. We have also assumed that the jet remains perfectly aligned with the deflector such that the momentum change is only in the horizontal direction. [7]

Note: This analysis is not the same as that of the Pelton wheel found in many textbooks. In this question the jet of water impinges on a single deflector. The deflector moves to the right at speed U and the jet of water between the nozzle and deflector lengthens. This jet contains kinetic energy. In contrast, the Pelton Wheel is a large wheel containing many deflectors. These deflectors arrive repeatedly at the nozzle, so the jet between the nozzle and the deflector is continually broken and its kinetic energy is extracted. In the analysis, the only difference is in the volumetric flowrate used in part (d). For this question, the volumetric flowrate is that into the moving control volume, $A_1(V_1-U)$. For the Pelton wheel analysis, the volumetric flowrate is that from the nozzle, A_1V_1.

2015 IA Mechanical Engineering Section B solutions

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Q7 Mixer arm

a) We can take pairs of semicircles on opposing sides together. The moment of inertia of a thin disc around its rotational axis is $MR^2/2$, so using the perpendicular axis theorem, the moment of inertia around one of its diameters (e.g. the mixer axis) is half of that, $MR^2/4$. The four paddles are equivalent to two discs (each of mass 2M), so altogether the moment of inertia is $I = MR^2$.

b) The final angular velocity is $\omega = 10 \times 2\pi \approx 63\ \mathrm{s}^{-1}$, so the final kinetic energy is $I\omega^2/2 = 0.1 \times 0.1^2 \times \frac{1}{2} \times 63^2 \approx 2$ Ws, this is the amount of work done over one second, so the average power is 2 W.

Q8 Track

a) Use conservation of energy. Assume the mass is $m = 1$, the initial kinetic energy of the ball is KE $= \frac{1}{2}v^2$. When the ball is at an angle θ, the potential energy has increased by PE $= (1 + \cos\theta)Rg$, and therefore the kinetic energy for the ball moving at speed $u(\theta)$ is KE $= \frac{1}{2}u^2 = \frac{1}{2}v^2 - (1 + \cos\theta)Rg$. So $u(\theta) = \sqrt{v^2 - 2(1 + \cos\theta)Rg}$.

b) The condition for not losing contact is that the force between the ball and the track should not drop below zero, when the ball would go into free fall. The place where this would happen first is when the centripetal acceleration and the gravitational acceleration point in the same direction, i.e. for $\theta = 0$. The centripetal acceleration is u^2/R, so the condition is $(v^2 - 2(1 + 1)Rg)/R = g$, which gives $v = \sqrt{5Rg}$.

Q9 Comet

a) Only the mass below the particle gives rise to a nonzero force, and as its mass is $\rho\frac{4}{3}\pi r^3$, its gravitational pull is $F(r) = \rho\frac{4}{3}\pi r^3 Gm/r^2 = \frac{4}{3}\rho\pi Gmr$.

b) The work done against gravity is

$$\int_r^R F(r)dr = \int_r^R \frac{4}{3}\rho\pi Gmrdr = \frac{4}{3}\rho\pi Gm[\frac{1}{2}r^2]_r^R = \frac{2}{3}\rho\pi Gm(R^2 - r^2).$$

c) The mass at radius r is $\rho\pi a^2 dr/4$, so the work in lifting all the mass up (from both sides of the comet) is

$$\int_0^R \frac{4}{3}G\rho^2\pi^2 a^2 \frac{1}{4}(R^2 - r^2)dr = \frac{2}{6}G(\rho\pi a)^2[R^2 r - r^3/3]_0^R = 2(\rho\pi a)^2 GR^3/9.$$

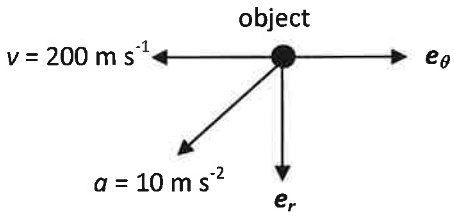
d) The force on a test particle is linear with r, so the particle behaves like a simple harmonic oscillator with force constant $k = \frac{4}{3}\rho\pi Gm$, so the frequency is $\sqrt{k/m} = \sqrt{\frac{4}{3}\rho\pi G}$.

Cribs – Part IA Mechanical Engineering

Q10.

(a) $\vec{v} = -200\hat{e}_\theta$

$$\vec{a} = \frac{10}{\sqrt{2}}\hat{e}_r - \frac{10}{\sqrt{2}}\hat{e}_\theta$$



The radial component of the acceleration can be calculated as below-

$$r\dot{\theta} = -200$$
$$\therefore \dot{\theta} = -0.2 \,\mathbf{rad\,s^{-1}}$$
$$\ddot{r} - r\dot{\theta}^2 = 7.07 \,\mathbf{ms^{-1}}$$
$$\therefore \ddot{r} = 47.07 \,\mathbf{ms^{-1}}$$

(b) If R is the instantaneous radius of curvature:

$$a_\perp = \frac{s^2}{R}$$
$$\therefore R = \frac{\sqrt{2} \times (200)^2}{10}$$
$$\therefore R = 5.66 \,\mathbf{km}$$

Q11.

(a) By inspection the mode shapes are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The corresponding natural frequencies are

$\omega_x = \sqrt{\dfrac{2k}{m}}$ and $\omega_y = 4\sqrt{\dfrac{k}{m}}$.

(b) If the initial conditions are given by: $x = 0,\ y = 0,\ \dot{x} = 0,\ \dot{y} = 1$ at time $t = 0$ then the general equation for the particle can be written as:

$$x = A\cos\omega_x t + B\sin\omega_x t$$
$$y = C\cos\omega_y t + D\sin\omega_y t$$

Plugging in the initial conditions we get: $A = B = C = 0$ and $D = \frac{1}{\omega_y} = \frac{1}{4}\sqrt{\frac{m}{k}}$ and therefore the subsequent motion is given by $\begin{bmatrix} 0 \\ \frac{1}{4}\sqrt{\frac{m}{k}}\sin 4\sqrt{\frac{k}{m}}t \end{bmatrix}$

Q12.

(a) $m\ddot{x} = -k(x-y) - k(x-y) - \lambda(\dot{x}-\dot{y}) - 2\lambda(\dot{x}-\dot{y})$

$$m\ddot{x} + 2k(x-y) + 3\lambda(\dot{x}-\dot{y}) = 0$$
$$m(\ddot{x}-\ddot{y}) + 2k(x-y) + 3\lambda(\dot{x}-\dot{y}) = -m\ddot{y}$$
$$\therefore m\ddot{z} + 3\lambda\dot{z} + 2kz = -m\ddot{y}$$

$$\therefore \alpha = 3,\ \beta = 2$$

(b) $m\ddot{z} + 3\lambda\dot{z} + 2kz = -m\ddot{y}$ ....(1)

Putting $y = Ye^{i\omega t}$, $z = Ze^{i\omega t}$ we get:

$$-m\omega^2 Z + 3\lambda i\omega Z + 2kZ = -m\omega^2 Y$$

$$Z = \frac{-m\omega^2 Y}{-m\omega^2 + 3\lambda i\omega + 2k}$$
$$Z = \frac{Y}{\left(1 - \frac{2k}{m\omega^2} - \frac{3\lambda i}{m\omega}\right)}$$
$$|Z| = \frac{Y}{\sqrt{\left(\left(1 - \frac{2k}{m\omega^2}\right)^2 + \left(\frac{3\lambda}{m\omega}\right)^2\right)}}$$
$$\phi = \tan^{-1}\left(\frac{3\lambda\omega}{2k - m\omega^2}\right)$$

For Y = 10 mm and ω = 5 rad s^{-1}

$$|Z| = \frac{10}{\sqrt{\left(\left(1 - \frac{200}{25}\right)^2 + \left(\frac{15}{5}\right)^2\right)}} = 1.31 \text{ mm.}$$

$$\phi = \tan^{-1}\left(\frac{3\times5\times5}{200-25}\right) \approx 0.4 \text{ rad.}$$

(c) From the CUED Mechanics databook Pg. 11 we can infer that $\zeta \approx 0.39$ to maximise the working range for the transducer.

$$\zeta = \frac{3\lambda}{2\sqrt{2km}} = 0.39$$

$\therefore \lambda = 3.67$ N s m^{-1}

From CUED Mechanics databook Pg. 10

$$\omega_r = \frac{\omega_n}{\sqrt{1-2\zeta^2}}$$

$\therefore \omega_r = 16.90$ rad/s

(d) Dividing equation (1) by m, we can see that this maps to case 4.5 in the CUED Mechanics databook, Pg. 7.

From the graph on Pg. 7, the maximum overshoot is estimated to $\approx \dfrac{0.61 \times 14.14}{200} = 43$ mm

Also the logarithmic decrement is given by $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.66$.

$$\therefore N\delta = \ln\frac{z_{\max}}{z_{fin}} = \ln(1000)$$

$$\therefore N = 2.6$$

which at 13 rad/s takes 1.25 seconds to settle to within 0.1% of the maximum value.