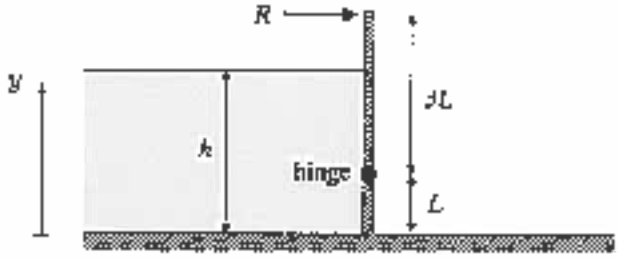
Pt IA paper 1 2016

Q1



(a) The force on the gate is the integral of the hydrostatic pressure on its surface. On the air side $p = p_a$, and on the water side $p = p_a + \rho g(h-y)$, so the resultant is

$$F = \int_0^h \rho g(h-y)\,\mathrm{d}A = \int_0^h \rho g(h-y) w\,\mathrm{d}y = \rho g w \int_0^h (h-y)\,\mathrm{d}y = \frac{1}{2}\rho g w h^2.$$

(b) The moment of R on the hinge is $M = 3RL$, and needs to balance the moment by the hydrostatic pressure, so

$$M = \int_0^h (L-y)\rho g(h-y)\,\mathrm{d}A = \rho g w \int_0^h (L-y)(h-y)\,\mathrm{d}y = \left(L - \frac{1}{3}h\right)\frac{1}{2}\rho g w h^2.$$

The above expression can also be obtained from the value of F and its point of action, $y = h/3$. The reaction must then be

$$R = \frac{1}{18}\rho g w h^2 \left(\frac{3L-h}{L}\right),$$

which is zero for $h = 0$ and $h = 3L$, positive for $0 < h < 3L$ and negative for $h > 3L$.

Q2

(a) Viscous effects are negligible between 1 and 2, so Bernoulli's equation can be applied. We then have

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2,$$

or

$$p_1 - p_2 = \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) = \frac{1}{2}\rho V_1^2\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right].$$

From continuity, $V_1 A_1 = V_2 A_2$, so substituting for V_2/V_1 we have

$$p_1 - p_2 = \frac{1}{2}\rho V_1^2 K_1,$$

with

$$K_1 = \left(\frac{A_1}{A_2}\right)^2 - 1.$$

(b) There is vigorous mixing and dissipation between 1 and 2, so Bernoulli's equation cannot be applied. Instead, the steady flow momentum equation can be applied to the flow in the pipe between sections 2 and 3. Neglecting friction forces at the walls we have

$$p_2 A_3 + \rho V_2 A_2 V_2 = p_3 A_3 + \rho V_3 A_3 V_3.$$

Note that p_2 acts on the whole section 2, including the back face of the expansion. From continuity, $V_2 A_2 = V_3 A_3$, but also $A_3 = A_1$ (and therefore $V_3 = V_1$), so substituting we have

$$p_2 A_1 + \rho V_1 A_1 V_2 = p_3 A_1 + \rho V_1 A_1 V_1,$$

which gives

$$p_2 - p_3 = \rho V_1^2\left(1 - \frac{V_2}{V_1}\right) = \rho V_1^2\left(1 - \frac{A_2}{A_1}\right) = \frac{1}{2}\rho V_1^2 K_2,$$

with

$$K_2 = 2\left(\frac{A_1}{A_2} - 1\right).$$

Q3

(a) From the streamline curvature,

$$\frac{dp}{dr} = \frac{\rho V^2}{r} = \rho\omega^2 r,$$

where r is the distance to the axis of symmetry. Integrating between A and B we have

$$\begin{aligned}\int_A^B dp &= \int_{r_A}^{r_B} \rho\omega^2 r dr,\\ p_B - p_A &= \rho\omega^2 \int_{r_A}^{r_B} r dr = \frac{1}{2}\rho\omega^2 r_B^2.\end{aligned}$$

(b) Let point A be at the free surface. Point B will then be under a column of water of height $\Delta h(r_B)$. The pressure at A is the ambient pressure,

$$p_A = p_a,$$

while the pressure at B will be

$$p_B = p_a + \rho g \Delta h(r_B).$$

The difference between them is $p_B - p_A = \rho g \Delta h(r_B)$, but must also satisfy the expression from (a), so we have

$$\rho g \Delta h(r_B) = \frac{1}{2}\rho\omega^2 r_B^2.$$

The shape of the free surface is then

$$\Delta h(r) = \frac{\omega^2 r^2}{2g}.$$

(c) The volume of water at rest is

$$V = \pi R^2 h_0,$$

and it must be the same when rotating. In the latter case, the volume can be calculated for instance as the sum of the volume of a cylinder of radius R and height $h_1 - \Delta h(R)$ and the volume under the paraboloid defined by $\Delta h(r)$ between $r = 0$ and $r = R$,

$$V = \pi R^2 (h_1 - \Delta h(R)) + \int_0^R \Delta h(r) 2\pi r dr.$$

Introducing the expression for $\Delta h(r)$ from (b), we have

$$V = \pi R^2 \left(h_1 - \frac{\omega^2 R^2}{2g}\right) + \int_0^R \frac{\omega^2 r^2}{2g} 2\pi r dr = \pi R^2 \left(h_1 - \frac{\omega^2 R^2}{4g}\right).$$

The relationship between h_1 and h_0 is therefore

$$h_1 = h_0 - \frac{1}{4}\frac{\omega^2 R^2}{g}.$$

The maximum ω is achieved for $h_1 = H$,

$$\omega_{max}^2 = \frac{4g}{R^2}(H - h_0).$$

(d) The shape of $\Delta h(r)$ remains unchanged, as follows from replacing part of the volume of water by an immersed solid –i.e. the cone. The solution $\Delta h(r)$ is only meaningful when the angle between the cone and the free surface at the contact point is positive, so

$$\left.\frac{d(\Delta h(r))}{dr}\right|_{r=h_1} \leq \tan(45^\circ) = 1.$$

Substituting for $\Delta h(r)$ from (b) we obtain

$$\frac{\omega^2 h_1}{g} \leq 1.$$

If ω increases beyond that point, the free surface is not stable and some liquid escapes running up the cone walls, until the remaining volume of liquid can satisfy the above condition.

Q4

(a) Applying isotropic relationships from the databook for $T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$, where
$T_1 = 300$ K, $p_1 = 1$ bar, $p_2 = 4.5$ bar and $\gamma = 1.4$
$T_2 = 300\ \text{K}\ (\frac{4.5}{1})^{0.4/1.4} = 461$ K

Intended solution - steady flow energy equation

Applying the first law of thermodynamics assuming steady state steady flow
$\cancel{\frac{dE_{cv}}{dt}} - \Sigma\left(\dot{m}_1\left(h_1 + \frac{V_1^2}{2} - \cancel{gz}\right)\right) + \Sigma\left(\dot{m}_2\left(h_2 + \frac{V_2^2}{2} + \cancel{gz}\right)\right) = \cancel{\dot{Q}} - \dot{W}$ and $\dot{m}_1 = \dot{m}_2$

$\dot{w} = h_2 - h_1 = c_p(T_1 - T_2) = 1.05$ kJ/(kg K) (300 K - 461 K) $\rightarrow$ $\boxed{\dot{w} = \text{-161.9 kJ/kg}}$

Alternative solution - constant volume

$\dot{w} = u_2 - u_1 = c_v(T_1 - T_2) = 0.718$ kJ/(kg K) (300 K - 461 K) $\rightarrow$ $\boxed{\dot{w} = \text{-115.6 kJ/kg}}$

(b) *Intended solution - steady flow energy equation*
The work is given as, $w_{12} = -200$ kJ/kg. Applying the first law of thermodynamics assuming steady state steady flow results in
$T_{2b} = T_1 - \frac{w_{12}}{c_p} = 300$ K + 200 kJ/kg /(1.005 kJ/(kg K))$\rightarrow$ $\boxed{T_{2b} = 499\ \text{K}}$

$\cancel{\frac{dS_{cv}}{dt}} - \Sigma[\dot{m}_1 s_1] + \Sigma[\dot{m}_2 s_2] = \int\frac{\cancel{\delta\dot{Q}}}{T} + \dot{S}_{irrev}$
$\dot{S}_{irrev} = s_2 - s_1$
From the databook for perfect gases the change in specific entropy is
$\Delta s_{12} = c_p \ln\left(\frac{T_{2b}}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 1.05$ kJ/(kg K) $\ln(\frac{499}{300})$ - 0.287 kJ/(kg K) $\ln(\frac{4.5}{1})$
$\boxed{\Delta s_{12} = 80\ \text{J/kg K}}$

Alternative solution - constant volume
$T_{2b} = T_1 - \frac{w_{12}}{c_v} = 300$ K + 200 kJ/kg /(0.718 kJ/(kg K))$\rightarrow$ $\boxed{T_{2b} = 578.6\ \text{K}}$

$\Delta s_{12} = c_p \ln\left(\frac{T_{2b}}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = 1.05$ kJ/(kg K) $\ln(\frac{578.6}{300})$ - 0.287 kJ/(kg K) $\ln(\frac{4.5}{1})$
$\boxed{\Delta s_{12} = 228\ \text{J/kg K}}$

Q5

(a) Calculate the mass of air within the balloon. Apply the ideal gas law to determine the mass in the balloon.
$m = \frac{p_{atm}V_1}{R_{air}T_1} = \frac{100{,}000\ \text{Pa}\ 0.001\ \text{m}^3}{287\ \text{J/(kg K)}\ 200\ \text{K}} \rightarrow \boxed{m = 0.0017\ \text{kg}}$

(b) To determine the final temperature, the first law of thermodynamics is applied

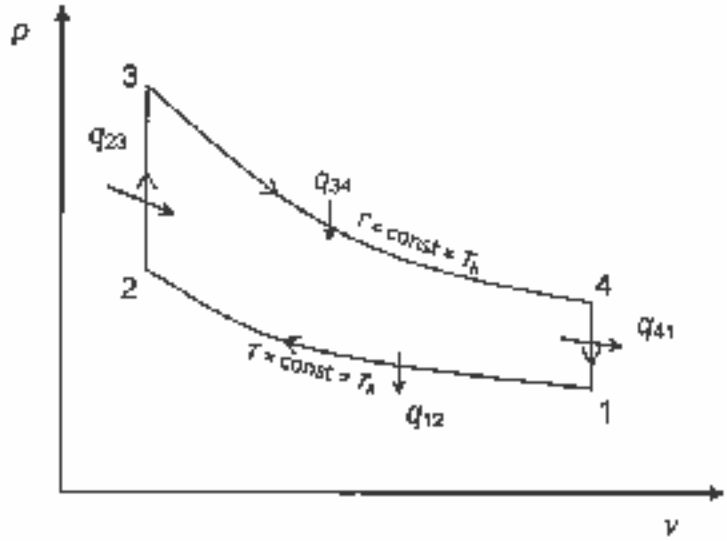
First Law $Q - W = \Delta U$
$\Delta U = mc_v\Delta T$
$W = p_{atm}\Delta V + \sigma\Delta A$
This formulation requires that ΔA be expressed in terms of ΔV
$A = 4\pi R^2$ and $V = \frac{4}{3}\pi R^3$, therefore $R = (\frac{3}{4\pi}V)^{1/3}$ so that $A = 4\pi(\frac{3}{4\pi}V)^{2/3}$

$T_2 = T_1 + \frac{1}{mc_v}(Q - p_{atm}(V_2 - V_1) - \sigma(4\pi(\frac{3}{4\pi}V_2)^{2/3} - 4\pi(\frac{3}{4\pi}V_1)^{2/3}))$

$T_2 = 300\ \text{K} + \frac{1}{0.0017\ \text{kg}\ 287\ \text{J/(kg K)}}(560\ \text{J} - 10^5\ \text{Pa}\ (0.005\ \text{m}^3 - 0.001\ \text{m}^3)$
$-100\ \text{N/m}(4\pi(3/(4\pi)0.005\ \text{m}^3)^{2/3} - 4\pi(3/(4\pi)0.001\ \text{m}^3)^{2/3}))$

$\boxed{T_2 = 320.4\ \text{K}}$

Q6



(a) $1 \rightarrow 2$, $q - w = \Delta u$, where $\Delta u = 0 \rightarrow q = w$ with $pdv < 0$ therefore $q < 0$. $\boxed{\text{Negative}}$
$2 \rightarrow 3$, $w = 0$, $q = \Delta u$, with $dT > 0$ therefore $q > 0$, $\boxed{\text{Positive}}$
$3 \rightarrow 4$, $q - w = \Delta u$, where $\Delta u = 0 \rightarrow q = w$ with $pdv > 0$ therefore $q > 0$, $\boxed{\text{Positive}}$
$4 \rightarrow 1$, $w = 0$, $q = \Delta u$, with $dT < 0$ therefore $q < 0$, $\boxed{\text{Negative}}$

(b) $r_v = v_4/v_3 = v_1/v_2$
$q_{in} = q_{23} + q_{34}$, where
$q_{23} = \Delta u = c_v \Delta T = c_v(T_h - T_l)$ and
$q_{34} = w_{34} = \int_3^4 pdv$
Using the ideal gas relation, $p = RT/v$, thus $q_{34} = \int_3^4 (RT/v)dv = RT_h \ln(v_4/v_3)$
$q_{34} = RT_h \ln(r_v)$
$q_{in} = c_v(T_h - T_l) + RT_h \ln(r_v) = (c_v + R\ln(r_v))T_h - (c_v)T_l$
By inspection, $\boxed{k_1 = (c_v + R\ln(r_v))}$ and $\boxed{k_2 = c_v}$.

(c) (i) $\eta = \frac{w_{net}}{q_{in}} = 1 - \frac{|q_{out}|}{q_{in}} = 1 - \frac{q_{12}+q_{41}}{q_{23}+q_{34}}$

$q_{12} = RT_l \ln(r_v^{-1}) = -RT_l \ln(r_v) = -287$ J/(kg K) 300 K $\ln(6) = -154.3$ kJ/kg
$q_{41} = c_v(T_l - T_h) = 718$ J/(kg K) (300 K - 1400 K)= -789.8 kJ/kg
$q_{23} = c_v(T_h - T_l) = 718$ J/(kg K) (1400 K - 300 K)= 789.8 kJ/kg
$q_{34} = RT_h \ln(r_v) = 287$ J/(kg K) 1400 K $\ln(6) = 719.9$ kJ/kg

$\eta = 1 - \frac{154.3+789.8}{789.8+719.9} = 0.375 \rightarrow \boxed{\eta = 37.5\%}$

(ii) With the regenerator the heat from q_{41} is used as an input to q_{23}. Therefore,

$\eta = 1 - \frac{|q_{12}|}{q_{34}} = 1 - \frac{154.3}{719.9} = 0.786 \rightarrow \boxed{\eta = 78.6\%}$ or
$\eta = 1 - \frac{RT_l \ln(r_v)}{RT_h \ln(r_v)} = 1 - \frac{T_l}{T_h} = 1 - \frac{300}{1400} = 0.786$

(d) To determine the heat transferred into the engine for $c_v = \alpha + \beta T + \gamma T^2$, where $\alpha = 610$ J/kg K, $\beta = 0.3$ J/kg K^2, $\gamma = 7 \times 10^{-5}$ J/kg K^3, requires $q_{in} = q_{23} + q_{34}$.
Here $q_{23} = \int_2^3 c_v dT = \int_2^3 (\alpha + \beta T + \gamma T^2)dT = \alpha(T_h - T_l) + \frac{\beta}{2}(T_h^2 - T_l^2) + \frac{\gamma}{3}(T_h^3 - T_l^3)$

$q_{23} = 610(1400 - 300) + \frac{0.3}{2}(1400^2 - 300^2) + \frac{7\times10^{-5}}{3}(1400^3 - 300^3) = 1014.9$ kJ/kg

From part (c), $q_{34} = 719.9$ kJ/kg

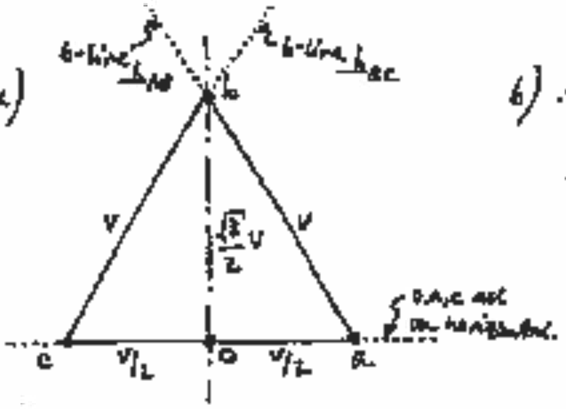
$q_{in} = 1014.9$ kJ/kg $+ 719.9$ kJ/kg $=$ $\boxed{1734.8 \text{ kJ/kg}}$

The efficiency is determined in the same manner as before
$\eta = 1 - \frac{q_{12} - q_{41}}{q_{23} + q_{34}}$

$q_{12} = -154.3$ kJ/kg
$q_{41} = -q_{23} = -1014.9$ kJ/kg
$q_{23} = 1014.9$ kJ/kg
$q_{34} = 719.9$ kJ/kg

$\eta = 1 - \frac{154.3 - 1014.98}{1014.9 + 719.9} = 0.326 \rightarrow \boxed{\eta = 32.6\%}$

7c)



behaves vertically like a structure, horizontally a mechanism

b) $\therefore V = \frac{mg}{2}$ by inspection of vertical equilibrium.

Then for H, use velocity diagram to match Power in = Power out.

$$\frac{\sqrt{3}}{2} mg v = 2\left(H \cdot \frac{v}{2}\right) \Rightarrow H = \frac{\sqrt{3}}{2} mg$$

8a) At max extension, both move at same speed v_1. Momentum conserved.

$2mV = (2m + m)v_1 \quad \therefore KE$ before $= \frac{1}{2}(2m)V^2 = mV^2$ Then consider energy balance

$v_1 = \frac{2V}{3} \Rightarrow KE$ after $= \frac{1}{2}(3m)\left(\frac{2V}{3}\right)^2 = \frac{2mV^2}{3} \quad \therefore$ PE in cable $= \frac{mV^2}{3}$

b) Cable is light wrt vehicles, but not massless

$\therefore$ EPE converted into KE of cable ends + some sound.

9a) $\underline{v_a} = \sqrt{2}\,\underline{i} - \sqrt{2}\,\underline{j} \quad \underline{v_b} = 2\underline{j} \quad \underline{v_c} = 2\underline{i}$

$\underline{a_a} = \left(\frac{2}{7} + \frac{1}{\sqrt{2}}\right)\underline{i} + \left(\frac{2}{7} - \frac{1}{\sqrt{2}}\right)\underline{j} \quad \underline{a_b} = 0 \quad \underline{a_c} = \frac{1}{2}\underline{i}$

These can all be written straightforwardly except $\underline{a_c}$ that requires consideration of the centripetal as well as tangential terms.

b) $\underline{V_{ac}} = \underline{v_a} - \underline{v_c} = (\sqrt{2} - 2)\underline{i} - \sqrt{2}\,\underline{j}$

$\underline{V_{bc}} = \underline{v_b} - \underline{v_c} = 2\underline{j} - 2\underline{i}$

c) $\underline{a_{ac}} = \underline{a_a} - \underline{a_c} = \left(\frac{1}{\sqrt{2}} - \frac{3}{14}\right)\underline{i} + \left(\frac{2}{7} - \frac{1}{\sqrt{2}}\right)\underline{j}$

Straight forward, provided terms are subtracted in the correct order and solutions to part (a) are correct.

d) Let speed of ball relative to B be d m/s and speed of ball relative to A be e m/s

$\underline{V_{ball}} - \underline{V_b} = d\,\underline{i} \qquad \underline{V_{ball}} - \underline{V_a} = e\left(\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}\right)$

(The key is to realise that A will catch the ball before she gets to C's starting point. Try sketching the velocity diagram.)

$\therefore d\,\underline{i} + \underline{V_b} = e\left(\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}\right) + \underline{V_a}$

$d\,\underline{i} + 2\underline{j} = e\left(\frac{1}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j}\right) + \sqrt{2}\,\underline{i} - \sqrt{2}\,\underline{j} \Rightarrow e = \sqrt{2}\left(\sqrt{2} + 2\right) = 4.83$ m/s

$\Rightarrow d = \left(\sqrt{2} + 2\right) + \sqrt{2} = 4.83$ m/s

Ball must cover 7m @ 4.83 m/s $\Rightarrow$ flight time = 1.45 s

$\therefore$ B moves $2 \times 1.45 = 2.90$ m

(i.e. B keeps pace with the ball in the $\underline{j}$ direction; from B's perspective, the ball approaches purely in $\underline{i}$ direction)

Q10

(a) $\Delta E = (T + V) - V_0$

$$= \left(\frac{1}{2} v^2 - \frac{M g}{r_0 + R} + \frac{MG}{R}\right) m$$

$$= \left(\frac{1}{2}(7900)^2 - \frac{6\times10^{24} \cdot 6.67\times10^{-11}}{(1200 + 6400)\times10^3} + \frac{6\times10^{24} \cdot 6.67\times10^{-11}}{6400\times10^3}\right) m$$

$$\Rightarrow 41.08\times10^6 \text{ J kg}^{-1}$$

With 10^{11} J for launch $m = \dfrac{1\times10^{11}}{41.08\times10^6}$

$$= \underline{2434 \text{ kg}}$$

(b) Use conservation of moment of momentum:

$$v_0 \cancel{m} r_0 = v_1 \cancel{m} r_1$$

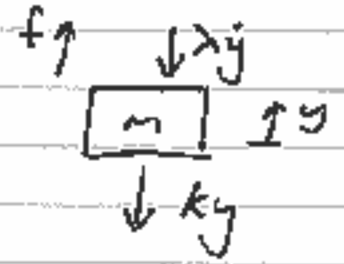
$$\therefore \quad r_1 = \frac{v_0 r_0}{v_1} = \frac{7900(1200 + 6400)\times10}{5431}$$

$$= 11{,}060 \text{ km}$$

The altitude therefore is 11060-6400 = 4660 km

Q. 11

(a)



free body diagram

$F = ma \longrightarrow \quad f - \lambda \dot{y} - ky = m\ddot{y}$

$\rightarrow \quad m\ddot{y} + \lambda \dot{y} + ky = f$

Standard form:

$$\frac{m}{k}\ddot{y} + \frac{\lambda}{k}\dot{y} + y = \frac{f}{k}$$

(b) $\left|\frac{X}{Y}\right| \leq 1.2$. From p. 9. of data book $\zeta \approx 0.48$ to keep $|X/Y| \leq 1.2$ @ $\omega = 0.8\omega_n$

Q 12

(a) "$F = ma$" on each mass

$$\underline{\underline{M}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \underline{\underline{C}} = \begin{bmatrix} 2\lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}$$

$$\underline{\underline{K}} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

(b) $\det(\underline{\underline{K}} - \underbrace{\lambda}_{=\omega_n^2} \underline{\underline{M}}) = 0$

$$\begin{vmatrix} 2k-\lambda m_1 & -k \\ -k & 2k-\lambda m_2 \end{vmatrix} = 0$$

$$(2k-\lambda m_1)(2k-\lambda m_2) - k^2 = 0$$

$$\Rightarrow \lambda_{1,2} = k\,\frac{m_1+m_2 \pm \sqrt{m_1^2+m_2^2-m_1m_2}}{m_1m_2}$$

$$\omega_1 = \sqrt{\lambda_1}, \quad \omega_2 = \sqrt{\lambda_2}$$

(c) $(\underline{\underline{K}} - \lambda_i \underline{\underline{M}})\underline{Y_i} = \underline{0}$

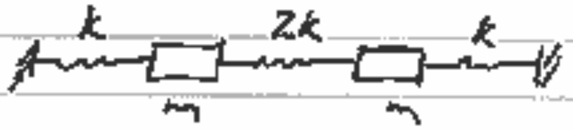
$$\begin{bmatrix} 2k-\lambda m_1 & -k \\ -k & 2k-\lambda m_2 \end{bmatrix}\begin{bmatrix} 1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2k - \lambda m_1 - kY_2 = 0$$

$$\Rightarrow Y_2 = \frac{2k-\lambda m_1}{k}, \quad Y = \begin{bmatrix} 1 \\ \dfrac{2k-\lambda m_1}{k} \end{bmatrix}$$

$$= 2 - \frac{m_1+m_2 \pm \sqrt{m_1^2+m_2^2-m_1m_2}}{m_2}$$

Q 12 (cont)

(d) 

From symmetry:

Case 1: $\underline{Y}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\omega_1 = \sqrt{\frac{k}{m}}$

Case 2: $\underline{Y}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ $\omega_2 = \sqrt{\frac{5k}{m}}$

$$y(t) = \underline{Y}_1(\alpha_1 \cos\omega_1 t + \beta_1 \sin\omega_1 t) + \underline{Y}_2(\alpha_2 \cos\omega_2 t + \beta_2 \sin\omega_2 t)$$

$t=0$: $\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\alpha_1 + \begin{bmatrix}1\\-1\end{bmatrix}\alpha_2$

$$\Rightarrow \alpha_1 = 1, \ \alpha_2 = 0$$

$\dot{y}$ @ $t=0$:

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}\beta_1\omega_1 + \begin{bmatrix}1\\-1\end{bmatrix}\beta_2\omega_2$$

$$\Rightarrow \beta_1 = \frac{1}{2\omega_1}, \ \beta_2 = -\frac{1}{2\omega_2}$$

$$y = \begin{bmatrix}1\\1\end{bmatrix}\left(\cos\omega_1 t + \frac{1}{2\omega_1}\sin\omega_1 t\right) - \begin{bmatrix}1\\-1\end{bmatrix}\frac{1}{2\omega_2}\sin\omega_2 t$$