Engineering Tripos, Part IA, 2019

Paper 1 Mechanical Engineering

Solutions

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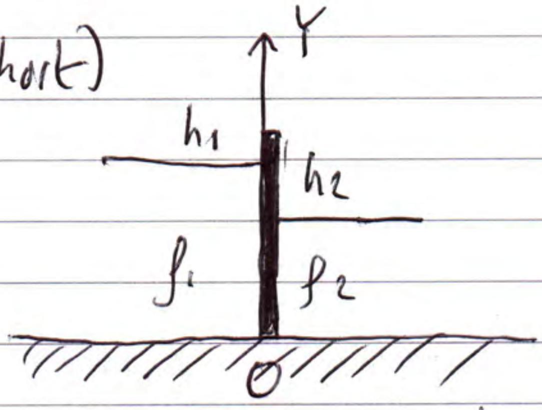
ENGINEERING TRIPOS PART IA Paper 1.

Section A

1. (short)

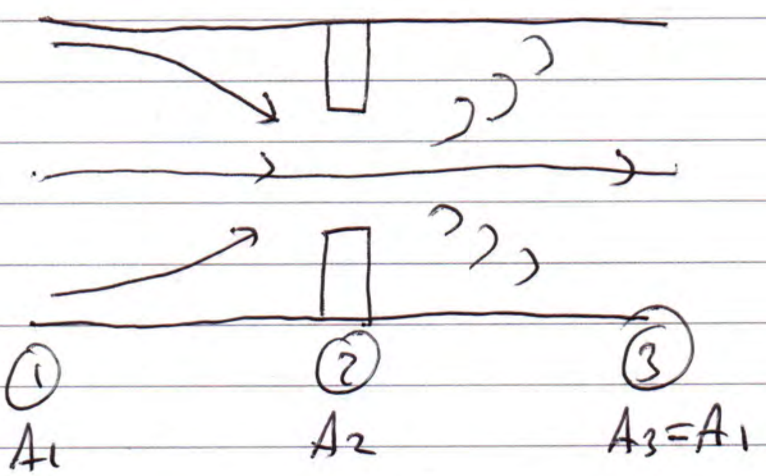


Moment about "O", $M_0 = \int_0^h \rho g (h-Y) Y \, dY = \rho g \left[\frac{hY^2}{2} - \frac{Y^3}{3} \right]_0^h = \frac{1}{6} \rho g h^3$

For zero moment: $\frac{1}{6} \rho_1 g h_1^3 = \frac{1}{6} \rho_2 g h_2^3$

$$\therefore \frac{h_1}{h_2} = \left(\frac{\rho_2}{\rho_1} \right)^{1/3}$$

2. (long)



(a) Bernoulli: $p_1 + \frac{1}{2}\rho u_1^2 = p_2 + \frac{1}{2}\rho u_2^2$

Continuity: $u_1 A_1 = u_2 A_2$

$$\therefore \frac{p_2 - p_1}{\frac{1}{2}\rho u_1^2} = 1 - \left(\frac{A_1}{A_2}\right)^2$$

(b) Momentum: $p_2 + \underbrace{\rho u_1}_{\dot{m}} u_2 = p_3 + \underbrace{\rho u_1}_{\dot{m}} u_3$ $\cancel{A_1}$

Continuity: $u_2 A_2 = u_3 A_3$

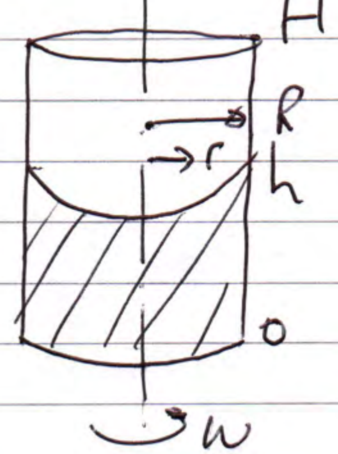
$$\therefore \frac{p_3 - p_2}{\frac{1}{2}\rho u_1^2} = 2\left(\left(\frac{A_1}{A_2}\right) - 1\right)$$

(c) ①–③ momentum: $(p_1 A_1 + \rho \cancel{A_1 u_1^2}) - (p_3 \overset{=A_1}{A_3} + \rho \cancel{A u_1 \overset{u_1}{u_3}}) = F$

$$\therefore F = A_1(p_1 - p_3) = p_1 - \left(p_1 + \frac{1}{2}\rho u_1^2\left(1 - \left(\frac{A_1}{A_2}\right)^2\right) - 2\frac{1}{2}\rho u_1^2\left(\frac{A_1}{A_2} - 1\right)\right)$$

$$\therefore F \Big/ \tfrac{1}{2}\rho u_1^2 A_1 = \left(\left(\frac{A_1}{A_2}\right) - 1\right)^2$$

3. (short)



$p = \rho g h$

(a) curved streamlines: $\frac{dp}{dr} = \rho \frac{v_\theta^2}{r}$; $v_\theta = \omega r$ ← forced vortex

$\therefore \rho g \frac{dh}{dr} = \rho r \omega^2 \quad \therefore \underline{h = h_0 + \frac{1}{2}\frac{\omega^2 r^2}{g}}$

(b) volume, $Q = \int_0^R 2\pi r \, dr \, h = 2\pi \int_0^R \left(h_0 r + \frac{\omega^2 r^3}{2g}\right) dr$

$= 2\pi \left[\frac{h_0 r^2}{2} + \frac{\omega^2 r^4}{8g}\right]_0^R = \pi R^2 \left(h_0 + \frac{\omega^2 R^2}{4g}\right)$

$\therefore \underline{h_0 = \frac{Q}{\pi R^2} - \frac{1}{4}\frac{\omega^2 R^2}{g}}$

(c) If depth @ R is H then $\frac{\omega^2 R^2}{2g} = H - h_0$ from (a)

so from (b) $\frac{Q}{\pi R^2} - h_0 = \frac{H - h_0}{2}$

$\therefore \underline{Q = \pi R^2 \left(\frac{H + h_0}{2}\right)}$

4

2 (a) $p = kv^{-n}$,

Expansion is fully resisted $\Rightarrow w = \int p dv = \int_1^2 kv^{-n} dv = \frac{k}{1-n}\left[v^{(1-n)}\right]_1^2 = \frac{p_1 v_1 - p_2 v_2}{n-1}$

$$v_1 = \frac{RT_1}{p_1} = \frac{287 \times 280}{4 \times 10^5} = 0.2009\, m^3 kg^{-1},$$

$$v_2 = \left(\frac{p_1}{p_2} v_1^n\right)^{\frac{1}{n}} = \left(4 \times 0.2009^{1.2}\right)^{\frac{1}{1.2}} = 0.6378\, m^3 kg^{-1}$$

$$\therefore w = \frac{4 \times 10^5 \times 0.2009 - 10^5 \times 0.6378}{0.2} = \underline{82.9 \text{ kJ kg}^{-1}}$$ [6]

(b) $T_2 = \frac{p_2 v_2}{R} = \frac{10^5 \times 0.6378}{287} = \underline{222 \text{ K}}$

Apply 1st law of thermodynamics: $q - w = \Delta u$

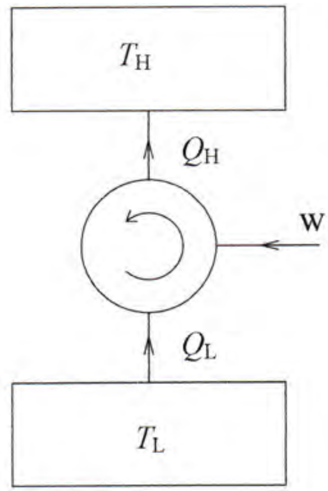
$$\therefore q = w + \Delta u = w + c_v \Delta T = 82.9 + 0.718 \times (222.2 - 280)$$

Heat transferred to the system per unit mass, $\underline{q = 41.4 \text{ kJ kg}^{-1}}$

Note that both q and w are +ve, but Δu is −ve. [4]

5

3 (a) The Clausius Inequality, $\oint \frac{dQ}{T} \le 0$



$$\therefore \frac{Q_L}{T_L} - \frac{Q_H}{T_H} \le 0$$

1st Law: $\frac{Q_L}{T_L} - \frac{W + Q_L}{T_H} \le 0$

$$\frac{Q_L}{W}\frac{1}{T_L} - \frac{1}{T_H}\left(1 + \frac{Q_L}{W}\right) \le 0$$

$$\frac{Q_L}{W}\left(\frac{1}{T_L} - \frac{1}{T_H}\right) \le \frac{1}{T_H}$$

$$\frac{Q_L}{W}\left(\frac{T_H - T_L}{T_L T_H}\right) \le \frac{1}{T_H}, \Rightarrow \frac{Q_L}{W} \le \frac{T_L}{T_H - T_L}$$

$$COP_R \le \frac{T_L}{T_H - T_L}$$

[6]

(b) $\frac{Q_L}{W} \le \frac{T_L}{T_H - T_L}$

$$\therefore W \ge \frac{Q_L\left(T_H - T_L\right)}{T_L} = \frac{0.4 \times \left(20 - (-10)\right)}{263.15} = 0.0456 = \underline{46\ \text{W}}$$

More power is required due to (i) irreversibility in the cycle and (ii) in practice, to allow for heat transfer at the cold and hot sides of the cycle $T_L < -10^{O}$C and $T_H > 20^{O}$C. [4]

Q6 (Long)

(a)(i) FALSE. For a perfect gas $\Delta u = m c_v \Delta T$ (NOT c_p) irrespective of how the process is conducted. [3]

(ii) TRUE. From the SFEE:

$$\underset{\uparrow\ \text{ADIABATIC}}{\cancel{q}} - \underset{\uparrow\ \text{NO SHAFT}}{\cancel{w_x}} = (h_2 + \tfrac{1}{2}\underset{\text{SMALL}}{\cancel{V_2^2}}) - (h_1 + \tfrac{1}{2}\underset{\text{SMALL}}{\cancel{V_1^2}})$$ [3]

$\therefore \quad h_2 \simeq h_1$

$\Rightarrow \quad T_2 = T_1$ (for an ideal gas).

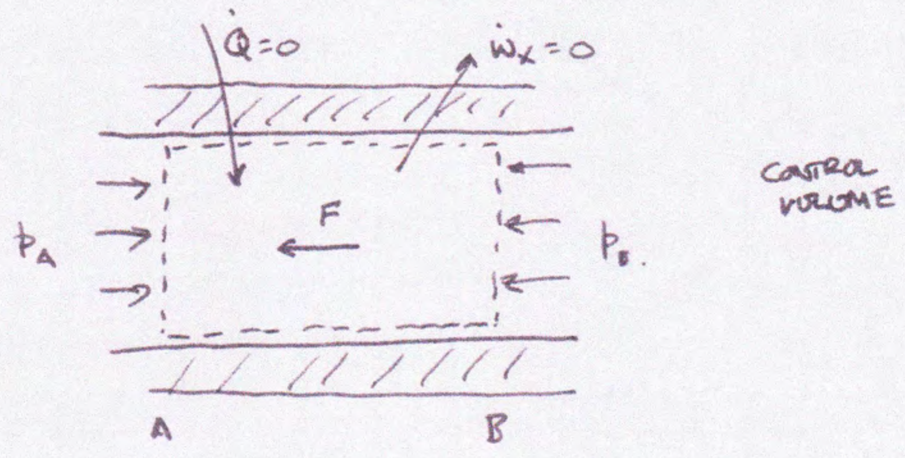
(iii) FALSE. The process is irreversible, but the flow might be losing heat s.t. $-\frac{\dot{Q}}{T} > \dot{S}_{irr}$ [3]

(b) (i) $\dot{m} = \rho v A = \frac{p_A}{RT_A} . V_A A$

$= \frac{2.5 \times 10^5}{287 \times 325} \times 400 \times 0.05$

$= \underline{\underline{53.6 \text{ kg s}^{-1}}}$ [3]

(ii)



Mass continuity:-

$$\rho_A U_A = \rho_B U_B$$

$$\therefore \quad \frac{p_A U_A}{T_A} = \frac{p_B U_B}{T_B} \quad \Rightarrow \quad T_B = \left(\frac{T_A p_B}{p_A U_A}\right) U_B$$

SFEE (with $q = 0$, $w_x = 0$):

$$c_p T_A + \tfrac{1}{2} U_A^2 = c_p T_B + \tfrac{1}{2} U_B^2$$

$$\therefore \quad \tfrac{1}{2} U_B^2 + c_p \left(\frac{T_A p_B}{p_A U_A}\right) U_B = c_p T_A + \tfrac{1}{2} U_A^2$$

$$\therefore \quad \tfrac{1}{2} U_B^2 + 1005 \times \left(\frac{325 \times 2}{2.5 \times 400}\right) U_B = 1005 \times 325 + \tfrac{1}{2} \times 400^2$$

$$\Rightarrow \quad U_B = \underline{\underline{460.3 \text{ ms}^{-1}}}$$

(other solution is negative and flow cannot change direction). [7]

(iii) The flow is adiabatic and so entropy must increase in the downstream direction. Assume $A \to B$

$$\Delta s = c_p \ln\left(\frac{T_B}{T_A}\right) - R \ln\left(\frac{p_B}{p_A}\right)$$

$$= c_p \ln\left(\frac{p_B U_B}{p_A U_A}\right) - R \ln\left(\frac{p_B}{p_A}\right)$$

$$= 1005 \times \ln\left(\frac{2 \times 460.3}{2.5 \times 400}\right) - 287 \times \ln\left(\frac{2}{2.5}\right) \simeq \underline{\underline{-19.1 \text{ J kg}^{-1}\text{K}^{-1}}}$$

$\therefore$ Flow is $\underline{\underline{\text{B TO A}}}$ (i.e., B is UPSTREAM) [6]

(iv) SFME: $(p_B - p_A) A + F = \dot{m} (u_A - u_B)$ (see C.V. figure for directions)

$$\therefore \quad F = (p_A - p_B) A + \dot{m} (u_A - u_B)$$

$$= 0.5 \times 10^5 \times 0.05 + 53.6 (400 - 460.3) = \underline{\underline{-0.732 \text{ kN}}}$$

This force is thus from A to B and since it is due to friction it must oppose the flow. Hence flow is $B \to A$. [5]

E2MD

Q7

(a) $$\frac{r_p}{r_A} = \frac{1-e}{1+e} = \frac{0.3}{1.7} \Rightarrow r_p = 2.12R$$

altitude at perigee $2.12R - R = 1.12R$

(b) $$v_p r_p = v_A r_A$$

$$\therefore v_p = \frac{1.7}{0.3} \times V \Rightarrow v_p = 5.67V$$

8 (**short**)

(a) Conservation of momentum: $Mu = MU + mV$
Conservation of energy: $Mu^2 = MU^2 + mV^2$
Solve for V:
Rearagne energy eqn: $M(u-U)(u+U) = mV^2$
Rearagne momentum eqn: $M(u-U) = mV$
Divide these two equations: $u+U = V$
Eliminate U in momentum equation: $M(2u-V) = mV$
Solve for V: $V = \frac{2Mu}{m+M}$.

(b) Applying the above rule twice, the final speed of the ball is
$V = \frac{2m_3 \frac{2Mu}{m_3+M}}{m_3+m} = \frac{4m_3 Mu}{(m+m_3)(M+m_3)}$.
We need to maximize

$$g = \frac{m_3}{(m+m_3)(M+m_3)}$$

Using the hint, we instead minimize 1/g:

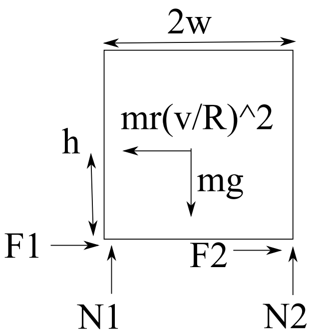
$$h = 1/g = \frac{Mm}{m_3} + m_3 + m + M.$$

$$\frac{dh}{dm_3} = -\frac{Mm}{m_3^2} + 1 = 0$$

which gives us $m_3 = \sqrt{Mm}$.

9 (**short**)

(a) Force diagram of the car from behind.



The force $mr(v/R)^2$ is a D'Alambert/centrifugal force. If the car goes too fast, this force will tip the car anti-clockwise, so the inner wheel (B) will leave the ground.

(b) Moments around lower left vertex:

$$m\frac{v^2}{R}h + N_2 2w = mgw.$$

At v_t we have $N_2 = 0$ so

$$m\frac{v^2}{R}h = mgw.$$

$$v_t = \sqrt{gR\frac{w}{h}}.$$

(c) Again moments around the lower left vertex:

$$m\frac{v^2}{R}h - mgw = I\dot{\omega}.$$

and putting in $v = 2v_t$

$$3mgw = I\dot{\omega}.$$

However, this I is around A, not around the center of mass, so we work it out using the parallel axis theorem:

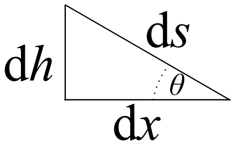
$$I = I_G + m\left(h^2 + w^2\right) = m(k^2 + h^2 + w^2)$$

Finally the angular acceleration is

$$\dot{\omega} = \frac{3gw}{k^2 + w^2 + h^2}.$$

10 (**long**)

(a) If we zoom in on an infinitesimal bit of track, we get a right-angled triangle:



Applying trig gives

$$\sin\theta = \frac{\mathrm{d}h}{\mathrm{d}s}.$$

(b) Conservation of energy:

$$\frac{1}{2}m\dot{s}^2 = mg\Delta h = mg(ks_0^4 - ks^4)$$

Solving for $\dot{s}$ we get

$$\dot{s} = \sqrt{2gk(s_0^4 - s^4)}.$$

(c) Applying $F = ma$ in the tangential direction:

$$mg\sin\theta = ma_{tangential}$$

Using the result from (a), this is

$$a_{tangential} = g\frac{\mathrm{d}h}{\mathrm{d}s} = 4gks^3.$$

(d) From the databook, normal acceleration is

$$a_{normal} = \dot{s}\dot{\theta} = \dot{s}^2\frac{d\theta}{ds}.$$

Using the $\sin\theta \approx \theta$ approximation, we have

$$\theta = \frac{\mathrm{d}h}{\mathrm{d}s} = 4ks^3.$$

Putting this into the above expression, we have,

$$a_{normal} = \dot{s}\dot{\theta} = \dot{s}^2 12ks^2 = 2gk(s_0^4 - s^4) \times 12ks^2 = 24gk^2(s_0^4 - s^4)s^2.$$

(e) Max velocity is at $s = 0$, where potential energy is lowest and kinetic energy is highest.

Max is tangential acceleration is at the start $s = s_0$, where the track is steepest.

To maximize normal acceleration we take the derivative wrt s

$$\frac{da_{normal}}{ds} = 24gk^2\left(2s_0^4 s - 6s^5\right) = 0$$

which we can solve to get $s = 0$ (where $a_{normal} = 0$ so this is a minimum) or

$$s^4 = \frac{1}{3}s_0^4 \quad \Rightarrow \quad s = \pm\left(\frac{1}{3}\right)^{1/4} s_0$$

(f) (i) The form for the velocity, $\dot{s} = \sqrt{2gk(s_0^4 - s^4)}$, only depends on g and k via the product gk. We could integrate this to find $s(t)$, which must also only depend on gk, so the time period T can also only depend on the product.

(ii) The relevant quantities are T, m g s_0 and k. The dimensions of each are

$$[m] = \mathrm{M}$$
$$[g] = \mathrm{LT}^{-2}$$
$$[s_0] = \mathrm{L}$$
$$[T] = \mathrm{T}$$
$$k = \frac{distance}{distance^4} \rightarrow [k] = \mathrm{L}^{-3}.$$

We are looking for a formula that relates T to the other four. However, we know T only depends on g and k via their product gk, so we really only have three:

$$[gk] = \mathrm{T}^{-2}\mathrm{L}^{-2},$$
$$[s_0] = \mathrm{L},$$
$$[m] = \mathrm{M}.$$

There is only one way to put these together and make something with units of time:

$$T \propto \sqrt{\frac{1}{s_0^2 gk}} = \frac{1}{s_0\sqrt{gk}}.$$

The ratio of time periods for different starting points is thus

$$\frac{T_a}{T_b} = \frac{s_b}{s_a},$$

and therefore it will take $T_a = T_b\frac{s_b}{s_a} = 1 \times \frac{1}{5} = 0.2\text{s}$ from the new starting point.

Q11

(a) By inspection natural frequencies are $\omega_1 = \sqrt{\frac{k}{J}}$ and $\omega_2 = \sqrt{\frac{3k}{J}}$ and normal modes are $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$.

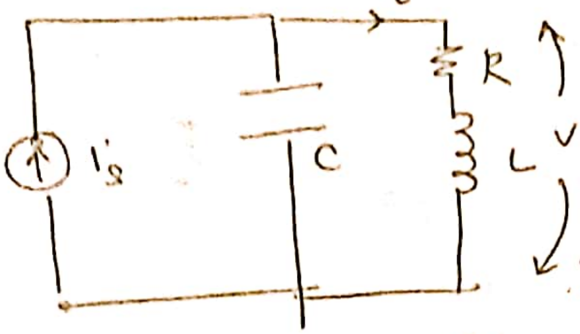
(b) At $t=0$, $\underline{\theta} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\dot{\underline{\theta}} = \begin{pmatrix}0\\0\end{pmatrix}$

$\therefore$ response involves both modes and by inspection / otherwise an equal amount of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ modes are excited.

$$\therefore \quad \theta_1 = \frac{1}{2}(\cos\omega_1 t + \cos\omega_2 t)$$

$$\theta_2 = \frac{1}{2}(\cos\omega_1 t - \cos\omega_2 t).$$

(a) Q12



$$i_s = C\frac{dv}{dt} + i$$

$$\therefore i_s = C\frac{d}{dt}\left[iR + L\frac{di}{dt}\right] + i$$

$$i_s = RC\frac{di}{dt} + LC\frac{d^2 i}{dt^2} + i$$

$$\therefore \frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC} i = \frac{1}{LC} i_s$$

(b) $\omega_n^2 = \frac{1}{LC} = 10^{12}$ $\therefore \omega_n = 10^6$ rad/s.

$2\beta\omega_n = \frac{R}{L} = 10^6 \Rightarrow \beta = 0.5$.

(c) From databook taking $\beta = 0.5$,

maximum value $\simeq 1.16$. (over all ω).

$\therefore$ maximum voltage $= 1.16 \times 0.01 \times 100 \simeq 1.16$ V

(d) maximum value if input is unit step function. Vmax ~~.~~ see databook for max. peak value $\simeq 1.17 \times$ nominal value.

$\therefore V_{max} = 100 \times 1.17 = 117$ volts.

(e) To limit excursion in voltage to 10% of maximum value $\beta = 0.6$ (from databook)

$$\therefore R = \frac{0.6}{0.5} \times 100 = 120\,\Omega.$$

Engineering Tripos Part IA 2019
Paper 1: Mechanical Engineering
Examiners comments

Question 1
Standard sluice gate hydrostatics; very well done – clearly well understood & rehearsed.

Question 2
Standard question, Bernouilli/continuity/momentum balance. Well understood & well done. There were several queries in the exam – and a number of notes in their scripts – about what "dynamic head" meant. This is a surprise but there was no disadvantage apparent – candidates simply asserted that $1/2\rho u^2$ must be "dynamic head"; some called it gH (which was not penalized)!

Question 3
Standard rotating can containing a liquid with a free surface. Less well understood (surprisingly as this is a familiar exam topic) and variable answers – candidates mostly got lost in (un-necessary) algebra.

Question 4
Short question on a piston in a cylinder with polytropic expansion. This is a fairly standard question. Most students tackled the problem by calculating the final volume by manipulating the polytropic equation with the equation of state, which is the right procedure. Some made algebraic mistakes here, but only a few made conceptual mistakes, e.g., using the wrong equation to calculate the work (neglecting the n-1 factor at the denominator for instance). Most students correctly applied the first principle of thermodynamics to calculate the heat transfer. Only a minority of students incorrectly calculated the final temperature.

Question 5
Short question on Clausius inequality and coefficient of performance (COP) for a cyclic refrigerator. This is a fairly standard question. The first part was purely mathematical (a proof) and the vast majority of students managed to solve it. Only very few did not manage to find the solution. The second part was an application of the definition of COP to find the power and explain why in a real cycle more power is required to extract heat from cold. This went down well on average.

Question 6
Long question divided in two parts. The first part was standard and theoretical. The majority of the students failed to recognize that the entropy of a flow does not always / necessarily increase (e.g., if there is cooling, entropy might decrease). The question in which students were asked to calculate the velocity of the flow had a vast spectrum of answers and approaches. The correct approach was to apply the energy equation for compressible flows. However, many students applied Bernoulli equation assuming an incompressible flow (arbitrary assumption, which is not correct); other students applied incorrectly the momentum equation. Most students recognize that to calculate the direction of the flow they had to calculate the entropy of the same. However, a few students made arguments based on the velocity of the flow or the energy. These arguments were incorrect. Overall, the impression is that many of the answers were either rushed or incomplete. Some students admitted (in writing) that they did not have time to complete the answer, thus, they just described what they would have done if they had had time. This suggests that many students were running out of time when tackling this question.

Question 7
This question was very well done as indicated by the high average mark. Most errors were either numerical or algebraic rather than conceptual.

Question 8
A high scoring question. The significant majority of candidates correctly analysed the single elastic collision in (a). Candidates were roughly equally split between invoking the coefficient of restitution vs explicit conservation of energy, and both approaches were successful. Most candidates also applied the result from (a) twice to find the velocity of the double collision in (b), but many candidates then failed to maximise this quantity correctly for full marks.

Question 9
A surprisingly high fraction of candidates misidentified how the car tilts in (a) (confusion over D'Alembert force vs acceleration?) but most nevertheless were easily able to obtain the critical velocity in (b) by taking moments. In (c), many candidates correctly calculated the total moment but only a few realized the need to use the parallel axis theorem before calculating the angular acceleration.

Question 10
Almost all candidates answered (a) and (b) (velocity) correctly. In (c) (tangential acceleration) some candidates resolved forces, but most differentiated their answer in (b). Similarly in (d) (normal acceleration) some candidates used v^2/r approaches, but most used (ds/dt)(d\theta/dt). In each case the second approach, though perfectly correct, lead to many errors because students failed to distinguish time and position derivatives. Part (e) was well answered by those who attempted it, though many students were hampered by having complicated incorrect expressions from (c) and (d). Part (f)(i) was very poorly answered, with most students searching for dimensional arguments rather than mechanical arguments. Many also misunderstood the question, and tried to argue that T(g,k,so)=T(g,k) rather than T(g, k, so)=T(gk, so). Students who understood the question then had little difficulty with f(ii), whilst many other made reasonable (if unsuccessful) efforts to apply dimensional analysis for which generous credit was given.

Question 11
Part (a) was generally well done though some wrote down incorrect expressions for the natural frequencies and normal modes "by inspection" with little justification. Part (b) was less well done with several students struggling to write down the general solution for the system response as a linear combination of the response defined by the normal modes. A subset of students who were able to get past this step made errors with working through the initial conditions though the final answer could have been arrived at by inspection.

Question 12
Some students had difficulty arriving at the second-order differential equation for resistor current in (a). Part (b) was generally well done though some students made numerical mistakes in their calculations. A subset of students failed to consult the databook / recognise a standard databook case for part (c). Those who did simply wrote down an expression for the magnitude of the current through the resistor, rather than both the magnitude and phase (relative to the input) of the voltage across the resistor as the question requested. Further errors were made in (c) in working through the frequency response graphs in the databook to arrive at the maximum voltage across the resistor at resonance. Students who attempted (d) identified the relevant page in the databook and worked through this correctly -- however, there were a few who read off the graphs incorrectly or simply wrote down a numerical value with little justification. Part (e) was generally well done by those who got this far with the nature of errors being similar to those seen in (d).