Engineering Tripos, Part IA, 2015

Paper 2 Structures and Materials

Solutions

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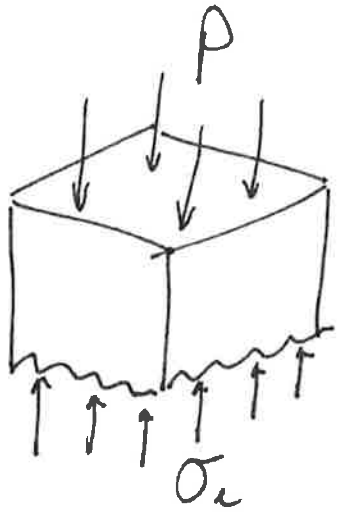
1. Assume $t \ll L$ so there is no variation of stress through the walls, there is no bending as the faces remain planar and the caps have no effect on the stress distribution in the walls.

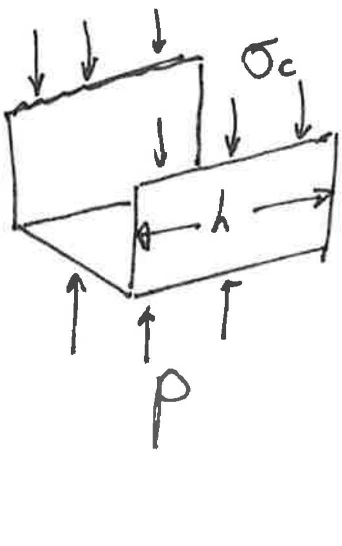
Then, using two free-body diagrams:



Using the usual convention that stresses are +ve if tensile:

$$-4Lt\,\sigma_L = P.L^2$$

$$\therefore \sigma_L = \frac{-PL}{4t}$$



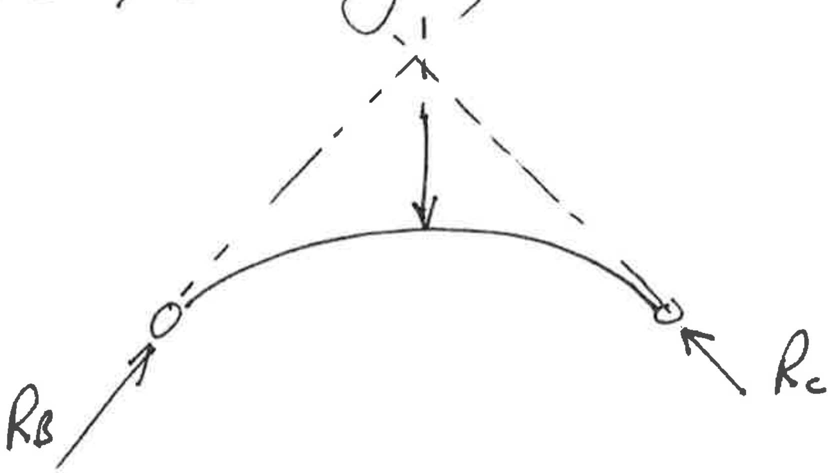
h is some arbitrary length, so

$$-2ht\sigma_c = PL.h$$

$$\sigma_c = -\frac{PL}{2t}$$

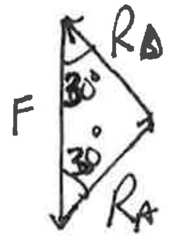
2. Only two forces act on AB, so they must be co-linear acting at A & B along AB.

Three forces act on BC : F, the reactions at B & C, so they must meet at a point :



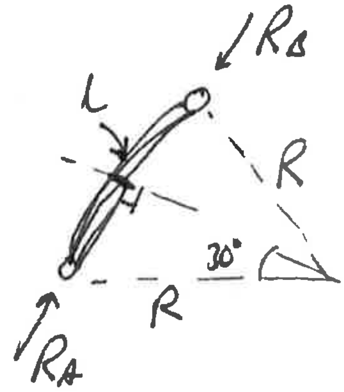
∴ the loading is symmetric, and there is no moment at B.

Force Polygon for whole structure :



$$\therefore \quad R_A \cos 30^\circ = \frac{F}{2} \; \therefore R_A = \frac{F}{\sqrt{3}}$$

Maximum moment in AB is at 30° :



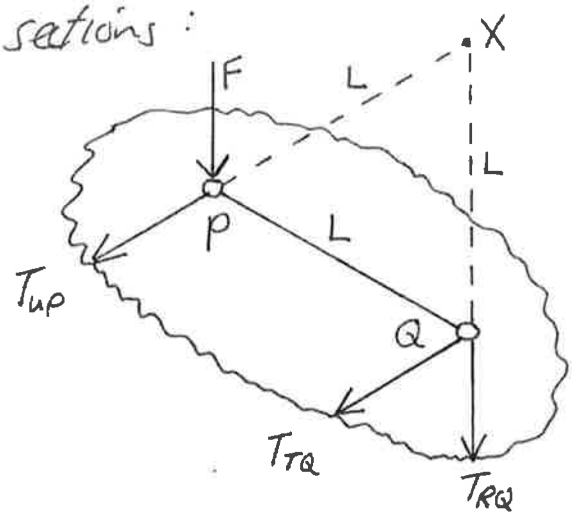
moment arm $l = R - R\cos 30^\circ$

$$= R\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$\therefore \text{max moment} = R_A . l$$

$$= FR\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)$$

3. Taking moments about S for the whole structure, there is no horizontal force at P, so the truss has both horizontal and vertical symmetry. We need therefore only find three bar forces, so using the method of sections:



Taking moments about Q: $T_{UP} . L . \frac{\sqrt{3}}{2} = -F . L \frac{\sqrt{3}}{2}$, $\therefore T_{UP} = -F$

about X: $T_{TQ} . L \frac{\sqrt{3}}{2} = F . L \frac{\sqrt{3}}{2}$, $\therefore T_{TQ} = +F$

about P: $T_{RQ} . L \frac{\sqrt{3}}{2} = -T_{TQ} . L \frac{\sqrt{3}}{2}$, $\therefore T_{RQ} = -F$

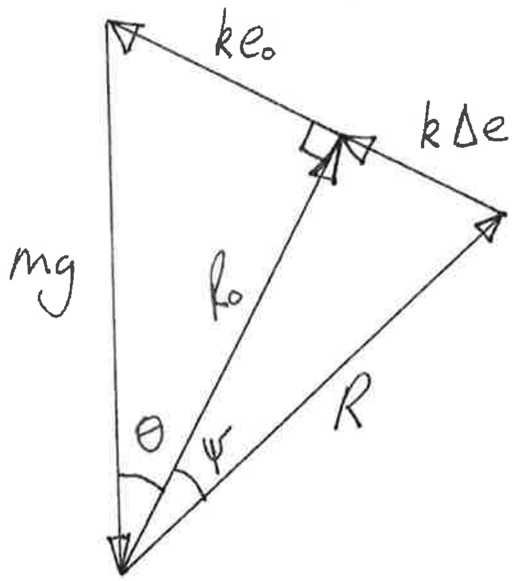
$\therefore$ By symmetry, the 6 perimeter bars of length L all experience compression $-F$, while the two interior bars of length 2L experience tension F.

$\therefore$ By <u>real</u> work: $F . \delta_P^{vert} = 6 . (-F)\left(\frac{-FL}{AE}\right) + 2(+F)\left(\frac{2FL}{AE}\right)$

$$\therefore \delta_P^{vert} = \frac{10FL}{AE} \text{ downwards}$$

4. In the absence of friction, $\psi = 0$, the spring extends by e_0, and the reaction force R_0 is perpendicular to the slope. When friction exists, the reaction force, R, rotates by ψ to drag the block down the slope, causing an additional spring extension Δe.

Both situations are shown in the force polygon:



$$\tan\theta = \frac{ke_0}{R_0}, \quad \tan\psi = \frac{k\Delta e}{R_0}$$

$$\therefore \tan\psi = \frac{\Delta e}{e_0}\tan\theta$$

5. (a) Taking first moments of area about the neutral axis, assuming all concrete below has cracked, that the bending stiffness of the steel bars about their own axis can be neglected, and transforming the steel to concrete:

$$\alpha L^2 . \frac{\alpha L}{2} + 2.\left(\frac{E_s}{E_c}\right) . \frac{\pi D^2}{4} . \left(\alpha L - 2D\right) =$$

$$2.\left(\frac{E_s}{E_c}\right) . \frac{\pi D^2}{4} . \left((1-\alpha)L - 2D\right)$$

$$\therefore \frac{L^3}{2} . \alpha^2 + 7 . \pi D^2 L . \alpha - \frac{14}{4} \pi D^2 . L = 0$$

$$\therefore \frac{1}{7\pi}\left(\frac{L}{D}\right)^2 \alpha^2 + 2\alpha - 1 = 0$$

$$6.55\alpha^2 + 2\alpha - 1 = 0$$

$$\therefore \alpha = \frac{-2 \pm 2\sqrt{1 + 6.55}}{2 \times 6.55} = \underline{0.267}$$

[or $\alpha L = 160$ mm]

(b) Making careful use of the parallel axis theorem, ignoring the bending of the bars around their own axes, and transforming to concrete:

$$I_c = 2\left(\frac{E_s}{E_c}\right)\cdot\frac{\pi\Delta^2}{4}\left[(160-100)^2 + (500-160)^2\right]$$

$$+ \frac{L.(\alpha L)^3}{12} + L.\alpha L.\left(\frac{\alpha L}{2}\right)^2$$

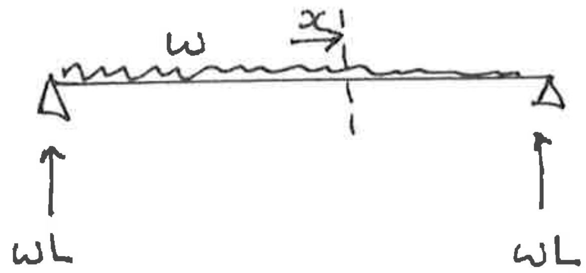
$$= \frac{14}{4}.\pi.\,2500\,(60^2 + 340^2) + \frac{600}{12}(160)^3 + 600.160.(80)^2$$

$$\therefore\ \underline{I_c = 4{,}100 \times 10^6\,mm^4}$$

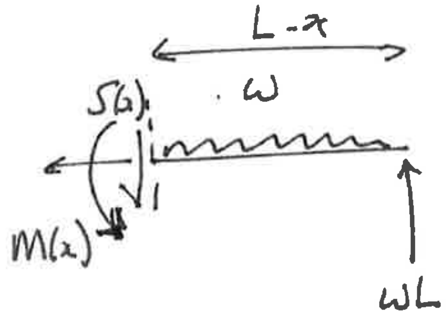
(c) Assume the lower bars yield:

$$M = \frac{\sigma I_c}{y} = \frac{350}{340} \times \frac{4{,}100\times10^6}{7} = \underline{600\ kNm}$$

6.



F.B.D at x:



$$S(x) = \omega L - \omega(L-x) = \omega x$$

$$M(x) = \frac{1}{2}\omega(L-x)^2 - \omega L(L-x) = \omega(L-x)\left[\tfrac{1}{2}(L-x) - L\right)$$

$$= -\frac{\omega}{2}(L-x)(L+x) = -\frac{\omega}{2}(L^2-x^2)$$

a). $I(x) = \dfrac{b\,d(x)^3}{12}$

$$\sigma_{max}(x) = \frac{M \cdot y_{max}}{I(x)} \quad \because \quad y_{max} = -\frac{d(x)}{2}$$

set $\sigma_{max}(x) = \sigma_y$ (yield)

$$\therefore \quad \sigma_y = \frac{3\omega(L^2-x^2)}{b\,d(x)^2}$$

$$\therefore \quad d(x) = \sqrt{\frac{3\omega}{b\sigma_y}(L^2-x^2}$$

b) For a cut across the beam, at the neutral axis, (8)

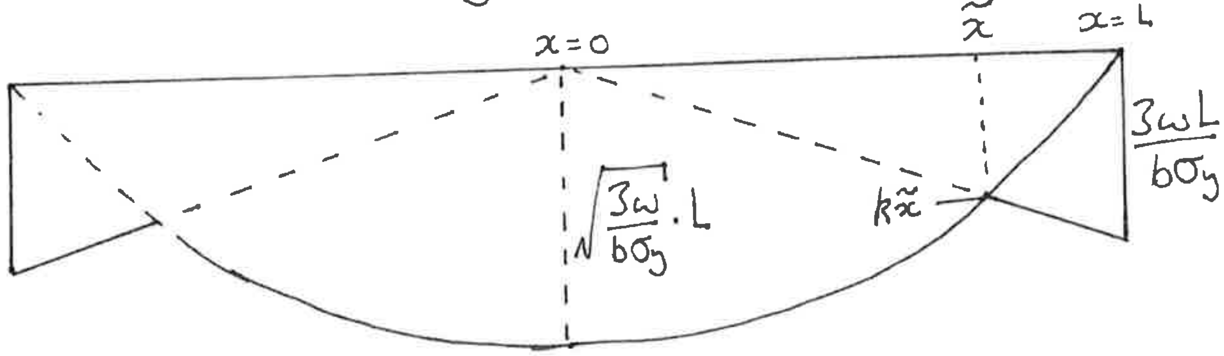
$$\tau(x) b = \frac{S(x) A_c(x) \bar{y}(x)}{I(x)} \quad \text{where } A_c(x) = \tfrac{1}{2} b\, d(x), \quad \bar{y}(x) = \tfrac{1}{4} d(x).$$

At the limit of yielding, $|\tau(x)| = \frac{1}{2}\sigma_y$

$$\therefore \quad \frac{\sigma_y}{2} = \left| \frac{\omega x}{b} \cdot \frac{12\, b\, d^2(x)}{8\, b\, d^3(x)} \right| = \frac{3 \omega |x|}{2 b\, d(x)}$$

$$\therefore \quad d(x) = \frac{3\omega |x|}{b \sigma_y}$$

c)



The two designs intersect where $k(L^2 - \tilde{x}^2) = k^2 \tilde{x}^2$, $k = \frac{3\omega}{b\sigma_y}$

$$\therefore \quad \tilde{x} = \frac{L}{\sqrt{1+k}}, \text{ and } d(\tilde{x}) = k\tilde{x} = \frac{kL}{\sqrt{1+k}}$$

The design should also account for:

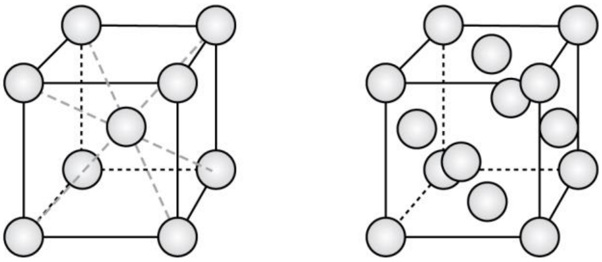
- the yield limits under combined shear and longitudinal stress
- stress concentrations at supports, load points or inflections
- other constraints that might limit acceptable depth variation
- manufacturability, fire protection & other practical issues.

J.M.A.
June 2015

SECTION B

7 (**short**)

(a)



BCC has

$$8\times\frac{1}{8}+1=1 \text{ corner atom + 1 centre atom}$$

$$\text{Atomic density}=\frac{2 \text{ atoms}}{\left(0.28\times10^{-9}\right)^3}=91.1\times10^{27} \text{ atoms m}^{-3}$$

FCC

$$\underset{\text{(corners)}}{8\times\frac{1}{8}} + \underset{\text{(faces)}}{6\times\frac{1}{2}}=4 \text{ atoms}$$

$$\text{Atomic density}=\frac{4 \text{ atoms}}{\left(0.35\times10^{-9}\right)^3}=93.3\times10^{27} \text{ atoms m}^{-3}$$

(b) atoms/m^3 $\propto$ density (increase in density on heating (BCC $\rightarrow$ FCC))
$\propto$ 1/volume (decrease in volume)

On heating

$$\frac{\Delta V}{V}=\frac{\left(93.3\times10^{27}\right)^{-1}-\left(91.1\times10^{27}\right)^{-1}}{\left(91.1\times10^{27}\right)^{-1}}=-0.0236$$

$$\therefore \frac{\Delta L}{L}\simeq\frac{1}{3}\frac{\Delta V}{V}=-0.79\%$$

Comments: This question was generally well treated, but the students should draw the crystal structures neatly to avoid mistakes when counting the number of atoms per unit cell. We did not ask to derive the relation given in part (b), but unrealistic results were severely penalized.

8 (**short**)

(a) *Hardness* is a measure of the resistance to indentation and is usually expressed as the load on the indenter divided by the projected area of indentation.

Yield stress is nominal stress at elastic limit, when yielding starts.

$\sigma_t = 450\ \varepsilon_t^{0.5}$ MPa

$\varepsilon_n = 0.25 \quad \therefore \varepsilon_t = \ln(1+\varepsilon_n) = \ln(1+0.25) = \ln 1.25 \simeq 0.223$

$\sigma_t = 450(\ln 1.25)^{0.5} \simeq 213$ MPa

This is the yield stress (for small strains, $\varepsilon_n \approx \varepsilon_t$ and $\sigma_n \approx \sigma_t$).

The hardness will be approximately

$H = 3\sigma_y = 3 \times 213 = 639$ MPa

(b) In compression, we will have the same final true stress and strain

$\sigma_t \simeq -213$ MPa

$\varepsilon_t \simeq -0.223$

$\varepsilon_t = \ln(1+\varepsilon_n) = -0.223$

$\therefore \varepsilon_n = -0.20$ i.e. 20% reduction

Since the yield stress is the same after compression and tension and also because volume is conserved, the ratio of the loads will be given by

$$\frac{F_c}{A_c} = \frac{F_t}{A_t} \quad \therefore \frac{F_c}{F_t} = \frac{A_c}{A_t} = \frac{l_t}{l_c} = \frac{1.25}{0.80} = 1.56$$

Comments: This question caused significant problems. The vast majority of students can properly define the yield stress, but fewer students can define hardness. Most students knew the difference between true and nominal strain (and stress), but we were expecting some justification when estimating the yield stress using the given constitutive equation. Part (b) has been attempted by a handful of students.

9 **(short)**

(a)

$$E_L = V_f E_f + (1-V_f)E_m = V_f n E_m + (1-V_f)E_m = E_m(1+(n-1)V_f)$$

(b)

$$E_T = \left[\frac{V_f}{E_f} + \frac{1-V_f}{E_m}\right]^{-1} = \left[\frac{V_f}{nE_m} + \frac{1-V_f}{E_m}\right]^{-1} = \left[\frac{V_f + n(1-V_f)}{nE_m}\right]^{-1} = \left[\frac{n-(n-1)V_f}{nE_m}\right]^{-1}$$

$$= \frac{nE_m}{n-(n-1)V_f}$$

(c)

$$\frac{E_L}{E_T}=\frac{E_m\left(1+(n-1)V_f\right)}{\left(\dfrac{nE_m}{\left(n-(n-1)V_f\right)}\right)}=\frac{\left(1+(n-1)V_f\right)\left(n-(n-1)V_f\right)}{n}=$$

$$=\frac{n+n(n-1)V_f-(n-1)V_f-(n-1)^2V_f{}^2}{n}$$

$$\frac{d\left(\dfrac{E_L}{E_T}\right)}{dV_f}=n(n-1)-(n-1)-2(n-1)^2V_f=0$$

$$\therefore V_f=\frac{n^2-2n+1}{2(n-1)^2}=\frac{(n-1)^2}{2(n-1)^2}=\frac{1}{2}$$

The maximum $\dfrac{E_L}{E_T}$ is independent of *n*.

Comments: Very easy question as indicated by the high average. Parts (a) and (b) were done very well. Loss of marks was in part (c). Several candidates made errors in algebra when estimating the E_L/E_T ratio, others did not differentiate with respect to V_f to find the maximum.

10 **(short)**

(a) For the hollow tube

$$\varphi^e_{Buckl}=\frac{12I}{A^2}=\frac{12\left(\dfrac{\pi}{4}\left(r_o{}^4-r_i{}^4\right)\right)}{\pi^2\left(r_o{}^2-r_i{}^2\right)^2}=\frac{3}{\pi}\frac{\left(r_o{}^2+r_i{}^2\right)}{\left(r_o{}^2-r_i{}^2\right)}==\frac{3}{\pi}\frac{\left((5t)^2+(3t)^2\right)}{\left((5t)^2-(3t)^2\right)}=\frac{3}{\pi}\frac{\left(25t^2+9t^2\right)}{\left(25t^2-9t^2\right)}=\frac{102}{16\pi}\approx 2$$

where r_o and r_i are the outer and inner diameter respectively.

(b) Strut mass

$$m=\rho AL$$

$$F=\frac{\pi^2EI}{L^2}=\frac{\pi^2E\phi^e_BA^2}{12L^2}$$

Eliminate the free variable *A* from the objective using the constraint

$$A=\frac{L}{\pi}\sqrt{\frac{12F}{E\phi^e_B}}\qquad \therefore m=\rho\frac{L^2}{\pi}\sqrt{\frac{12F}{E\phi^e_B}}=\frac{\rho}{\sqrt{E\phi^e_B}}\sqrt{12F}L^2$$

Material	Young's modulus (GPa)	Density ($Mg\ m^{-3}$)	ϕ_B^e	$M = \frac{\sqrt{E\phi_B^e}}{\rho}\left(\frac{\sqrt{GPa}}{Mg\ m^{-3}}\right)$
Steel	210	7.8	2 (circular tube)	2.63
Wood	20	0.8	1 (square)	5.59

Square wood would be best.

Comments: This question was also done very well. In part (a), a handful of students assumed the strut had a thin-walled circular section; despite the fact the question states clearly it is a thick-walled one. Some candidates used the wall thickness as the inner diameter. Part (b) was done very well, but interestingly a very large number of candidates when comparing the steel tube with the wooden square, they focused entirely on the $E^{1/2}/\rho$ ratio and didn't use the shape factors ϕ_B^e for the tube (≈2 - from Part (a)) and that of the square (=1).

11 **(long)**

(a) (i) At failure

$$K_{max} = \frac{2}{\pi}\sigma_{max}\sqrt{\pi a} = K_{IC}$$

$$\therefore \sigma_{max} = \frac{30\cdot\pi}{2\sqrt{\pi\cdot 5\times10^{-3}}} = 376 \text{ MPa}$$

$$\Delta\sigma = \sigma_{max} - \sigma_{min} = \sigma_{max} - 0.1\sigma_{max} = 0.9\sigma_{max} = 0.9\cdot 376 = 338.4 \text{ MPa}$$

(ii)

$$\frac{da}{dN} = 4\times10^{-11}\Delta K^5 = 4\times10^{-11}\cdot\left(\frac{2}{\pi}\cdot 338.4\cdot\pi^{1/2}\cdot a^{1/2}\right)^5$$

$$= 4\times10^{-11}\cdot\left(\frac{676.8}{\pi^{1/2}}\right)^5\cdot a^{5/2} =$$

$$= 325\ a^{5/2}$$

$$\int_{a_o}^{a_f}\frac{da}{a^{5/2}} = 325\int_0^{N_f} dN$$

$$\therefore \left[-\frac{2}{3}a^{-3/2}\right]_{a_o}^{a_f} = 325\ N_f \Rightarrow \left[-\frac{2}{3}a^{-3/2}\right]_{a_o}^{5\times10^{-3}} = 325\cdot10^4$$

$$a_o^{-3/2} = 325\cdot10^4\cdot\frac{3}{2} + \left(5\times10^{-3}\right)^{-3/2} = 49\times10^5$$

$$\therefore\ a_o = 35\ \mu\text{m}$$

Therefore the initial crack length is 2×35 = 70 μm.

(b)

$$P_S(V) = \exp\left\{ \int_V -\left(\frac{\sigma}{\sigma_0}\right)^m \frac{dV}{V_0} \right\}$$

$$\sigma_{max} = \rho\omega^2 R L_1 = 3200 \cdot 3000^2 \cdot 200\times10^{-3} \cdot 30\times10^{-3} \text{ kg m}^{-1}\text{s}^{-2} = 172.8 \text{ MPa}$$

$$\ln P_s(V_1) = -\frac{1}{V_0}\int_0^{L_1}\left(\frac{\rho\omega^2 R x}{\sigma_0}\right)^m A_1 dx = -\frac{A_1}{V_0}\int_0^{L_1}\left(\frac{\rho\omega^2 R}{\sigma_0}\right)^m x^m dx = -\frac{A_1}{V_0}\left(\frac{\rho\omega^2 R}{\sigma_0}\right)^m \int_0^{L_1} x^m dx$$

$$= -\frac{A_1}{V_0}\left(\frac{\rho\omega^2 R}{\sigma_0}\right)^m \left.\frac{x^{m+1}}{m+1}\right|_0^{L_1} = -\frac{A_1}{V_0}\left(\frac{\rho\omega^2 R}{\sigma_0}\right)^m \frac{L_1^{m+1}}{m+1}$$

$$\therefore \ln\left(1-10^{-7}\right) = -\frac{A_1 L_1}{V_0}\left(\frac{\rho\omega^2 R L_1}{\sigma_0}\right)^m \frac{1}{m+1} = -\frac{A_1 L_1}{V_0}\left(\frac{\sigma_{max}}{\sigma_0}\right)^m \frac{1}{m+1} \quad (1)$$

$$\ln P_s(V_2) = -\left(\frac{\sigma}{\sigma_0}\right)^m \frac{V}{V_0} = -\left(\frac{\sigma}{\sigma_0}\right)^m \frac{A_2 L_2}{V_0} = -\frac{A_2 L_2}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m$$

$$\therefore \ln\left(1-10^{-2}\right) = -\frac{A_2 L_2}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m \quad (2)$$

$$\frac{(2)}{(1)} \quad \frac{\ln\left(1-10^{-2}\right)}{\ln\left(1-10^{-7}\right)} = \frac{A_2 L_2}{A_1 L_1}(m+1)\left(\frac{\sigma}{172.8}\right)^m = \frac{16\times10^{-6}\cdot 0.04}{20\times10^{-6}\cdot 0.03}\cdot 11 \cdot\left(\frac{\sigma}{172.8}\right)^m$$

$$10^5 = \frac{16\cdot 0.04}{20\cdot 0.03}\cdot 11\cdot\left(\frac{\sigma}{172.8}\right)^m$$

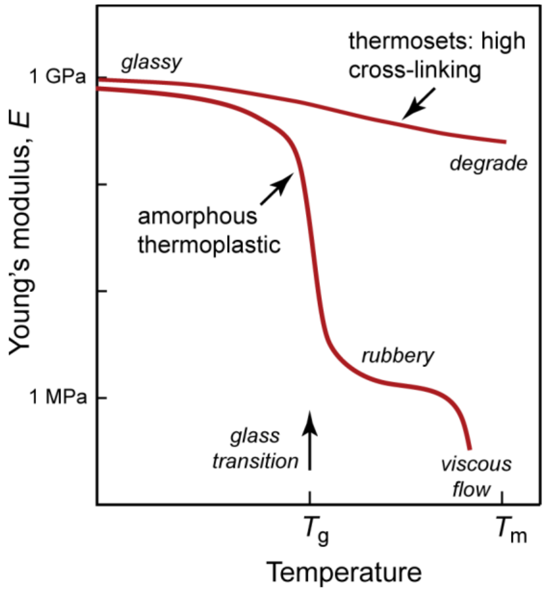
$$\therefore \sigma = 172.8\left(10^5\cdot\frac{20\cdot 0.03}{16\cdot 0.04\cdot 11}\right)^{1/10} = 427 \text{ MPa}$$

Comments: Parts (a) and (b) were generally done well. The main problem was numerical errors, which were not severely penalised. In part (a), several candidates didn't realise that by setting $K = K_{IC}$ the stress obtained was σ_{max}. In part (b), several candidates used σ_{max} instead of $\Delta\sigma$. Part (c) was answered less well. A very large number of candidates used the Weibull formula for varying stress for both the blade and the tensile test sample. A very large number of candidates used in their calculations the failure probability (P_f) value without converting it to a survival probability P_s ($=1-P_f$).

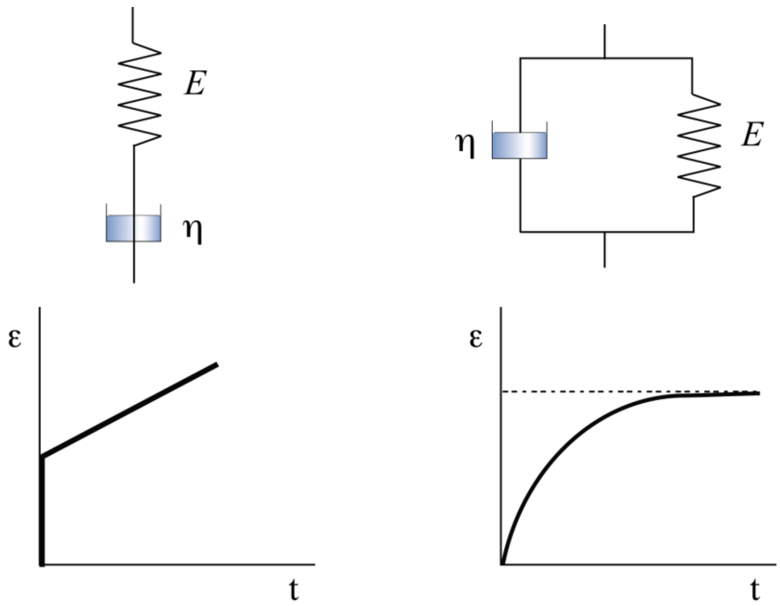
12 **(long)**

(a) A thermoplastic is a linear polymer which softens, becomes plastic and ultimately melts at higher temperature (hence the name). Examples are polyethylene (low & high density: LDPE, HDPE), polyvinylchloride (PVC), polypropylene (PP), polystyrene (PS). A thermoset is a cross-linked polymer which becomes less stiff as the temperature is raised, but which does not melt. Examples are epoxies, phenolics, polyurethane.

Amorphous thermoplastic: At $T < T_g$ (=glass transition temperature), *glassy region*. At $T \approx T_g$ segments of chain melt, and modulus falls steeply. At $T > T_g$, the modulus decreases by a factor of 1000. This is the *rubbery region*, in which the modulus is determined by the entanglement points. Above $1.4T_g$, the chains slip and the polymer melts to a viscous liquid (*viscous flow region*).
Thermoset: It is highly cross-linked, so there is no effect of T_g. Stiffer than thermoplastics in the glassy region. Young's modulus falls slowly on heating.



(b) Hookean springs are used to simulate the elastic response in polymer deformation: $\sigma = E\varepsilon$, where E is the elastic constant of the spring. Springs store energy and respond instantaneously.
Dashpots are used to model the viscous (time-dependent) response in polymer deformation: $\sigma = \eta\dot{\varepsilon}$, where η is viscosity and $\dot{\varepsilon}$ is the strain rate. Dashpots dissipate energy in the form of heat, and characterise the retarded nature of the response.



Spring and dashpot in series: The first part is the instantaneous elastic response while the second part is the viscous retarded response.

Spring and dashpot in parallel: Initially, on applying a step in stress, the dashpot "locks up" and prevents spring from extending. The strain reaches a limiting value at very long times (retarded elastic behaviour).

(c)

The strain ε is the sum of the displacement in the spring 1 and the viscous strain in the dashpot

$$\varepsilon(t) = \varepsilon_{spring1}(t) + \varepsilon_{viscous}(t)$$

$$\varepsilon(t) = \frac{\sigma}{E_1} + \varepsilon_{viscous}(t)$$

where $\sigma(t)$ is the stress carried by the spring.

Since the stress within the spring is also that on the dashpot (and equal to the imposed stress), we can differentiate the previous equation with respect to time

$$\dot{\varepsilon}(t) = \frac{\dot{\sigma}}{E_1} + \dot{\varepsilon}_{viscous}$$

$$\therefore E_1\dot{\varepsilon} = \dot{\sigma} + \frac{E_1}{\eta}\sigma$$

Assume harmonic response

$$\sigma = \hat{\sigma}e^{i\omega t}, \ \varepsilon = \hat{\varepsilon}e^{i\omega t}$$

Substitute in the governing equation to get

$$i\omega E_1\hat{\varepsilon} = \left(i\omega + \frac{E_1}{\eta}\right)\hat{\sigma}$$

$$\frac{\hat{\varepsilon}}{\hat{\sigma}} = \frac{\left(i\omega + \frac{E_1}{\eta}\right)}{i\omega E_1}$$

(d) "Resistance":

Spring 1: $\frac{1}{E_1}$, Dashpot: $\frac{1}{i\omega\eta}$, Spring 2: $\frac{1}{E_2}$

First branch (spring 1 and dashpot in series):

$$\frac{1}{E_1}+\frac{1}{i\omega\eta}=\frac{E_1+i\omega\eta}{E_1 i\omega\eta}$$

Total "resistance"

$$\frac{1}{1/E_2}+\frac{1}{\left(\frac{E_1+i\omega\eta}{E_1 i\omega\eta}\right)}=E_2+\frac{E_1 i\omega\eta}{E_1+i\omega\eta}=\frac{E_2(E_1+i\omega\eta)+E_1 i\omega\eta}{E_1+i\omega\eta}$$

$$\therefore \frac{\hat{\varepsilon}}{\hat{\sigma}}=\frac{E_1+i\omega\eta}{E_2(E_1+i\omega\eta)+E_1 i\omega\eta}$$

Comments: The students did well on this problem. The linear models of viscoelasticity seem well understood, although simple algebra was often poorly conducted. Answers for part (a) were very heterogeneous. To get the full mark in this part, all regimes of the graphs given in the lectures had to be presented and explained.