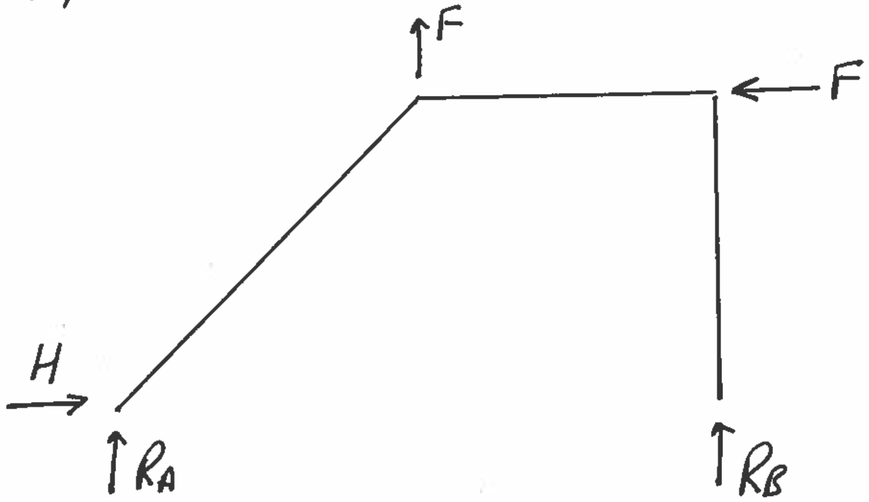
Engineering Tripos Part 1a 2016 Paper 2: CRIB

1: *Many candidates gave perfect answers to this question. Of those that didn't, many failed to calculate the reactions carefully, and the most common error was to have non zero bending moments in the right side vertical section of the frame. Because the support at B is on wheels, there can be no horizontal reaction at B, so therefore the bending moment all along the vertical leg ending at B must be zero.*

First, find the reactions:

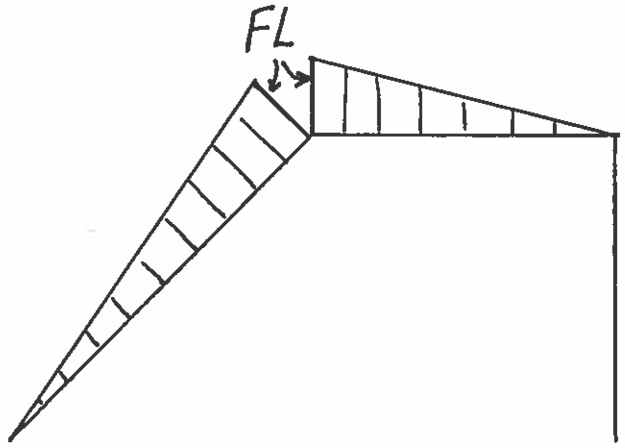


By horizontal equilibrium, $H = F$

By moments about A, $2LR_B = -2FL \therefore R_B = -F$ (downwards)

By moments about B, $2LR_A + FL = FL \therefore R_A = 0$

$\therefore$ Bending moment diagram is:



2: *Most candidates correctly used the formulae for the second moment of area of a rectangular section and the parallel axis theorem. The fact that t was much smaller than B means that the second moment of area of the two flanges about their own axes could be discarded, but the second moment of area of the web was still important, as was the second term of the parallel axis theorem for all three components. Candidates who worried about whether the flanges were separated by B, B+t, or B+2t, made their lives harder, and often made algebraic errors. Because t is small compared to B, this distance could be treated as B from the outset. Separately, it is much easier to find the centroid by taking first moments of area about the bottom of the beam rather than taking them about the centroid itself.*

If the Neutral Axis is y above the base, then taking first moments of area about the base, & using $t \ll B$

$$Bt \times B + Bt \times \tfrac{1}{2}B = (Bt + Bt + 2Bt) \times y$$

$$\therefore y = \frac{3}{8}B$$

Using the parallel axis theorem and ignoring the second moment of area of the flanges about their own axes,

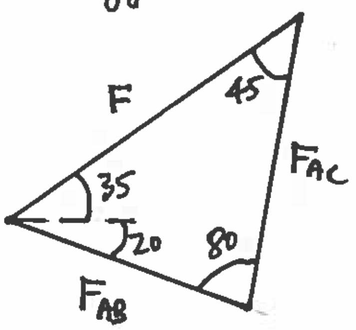
$$I = \underbrace{Bt(B-y)^2}_{\text{upper flange}} + \underbrace{\left[\frac{tB^3}{12} + Bt\left(\frac{B}{2}-y\right)^2\right]}_{\text{web}} + \underbrace{2Bt \cdot y^2}_{\text{lower flange}}$$

$$\therefore I = \frac{25}{64}B^3t + \frac{1}{12}B^3t + \frac{1}{64}B^3t + \frac{18}{64}B^3t$$

$$= \frac{37}{48}B^3t$$

3. What is ambiguous about "use GRAPHICAL METHODS ONLY"? *Many candidates tried to calculate the forces in the members by writing down equilibrium equations at A, and most then got the arithmetic wrong, or they took the cosine of the force F in the directions AB and AC, which is nonsense. Many took no account of the fact that one of the members is twice as long as the other so they got the extensions wrong. Many candidates drew diagrams the size of postage stamps and then measured lengths to the nearest centimetre. By all means make a sketch but then draw a diagram to scale as large as possible to maximise accuracy. Quite a lot took no account of the sense of the displacement - **it matters!** Most used rulers and some used protractors but it would have been a lot easier to draw both the displacement diagram and the force polygon on graph paper, which was provided. Freehand sketches got zero marks. It should be possible to get better than 1% accuracy by drawing although the paper was marked more leniently.*

Sketch Force Polygon (N.T.S)



Accurate version on next page →

From figure

Forces

$F_{AB} = 718$ N

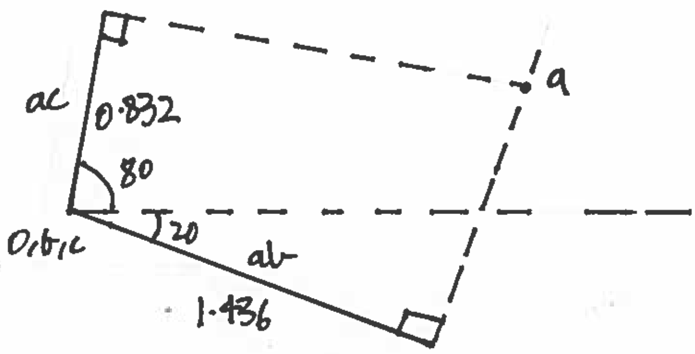
$F_{AC} = 832$ N

Extension AB $\quad e_{AB} = 2 . 718 \cdot \frac{L}{AE} = 1.436$ mm

N.B

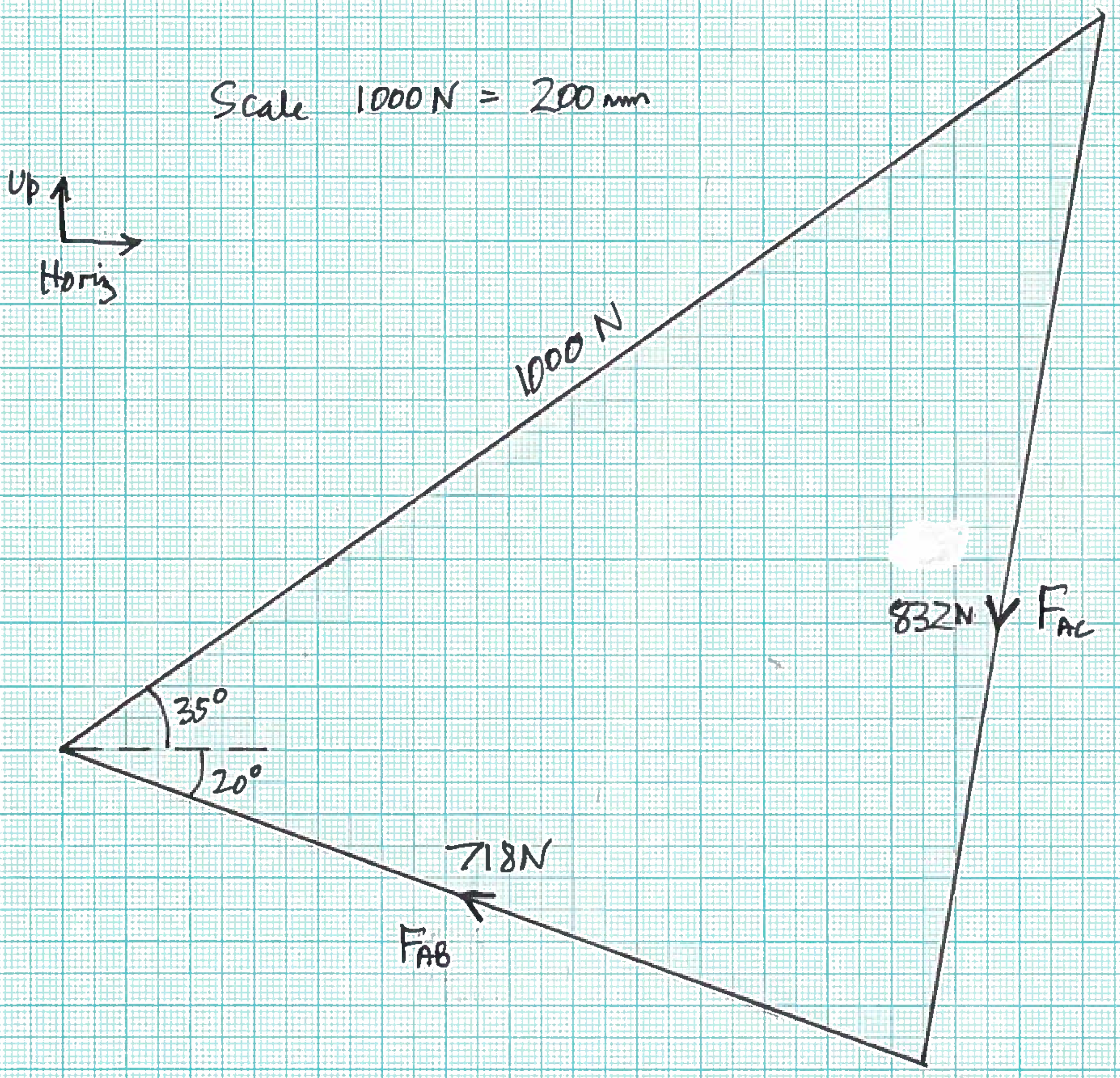
$e_{AC} = 832 \frac{L}{AE} = 0.832$ mm

Sketch Disp Diag (n.t.s)



Accurate plot later →

3. Polygon of forces



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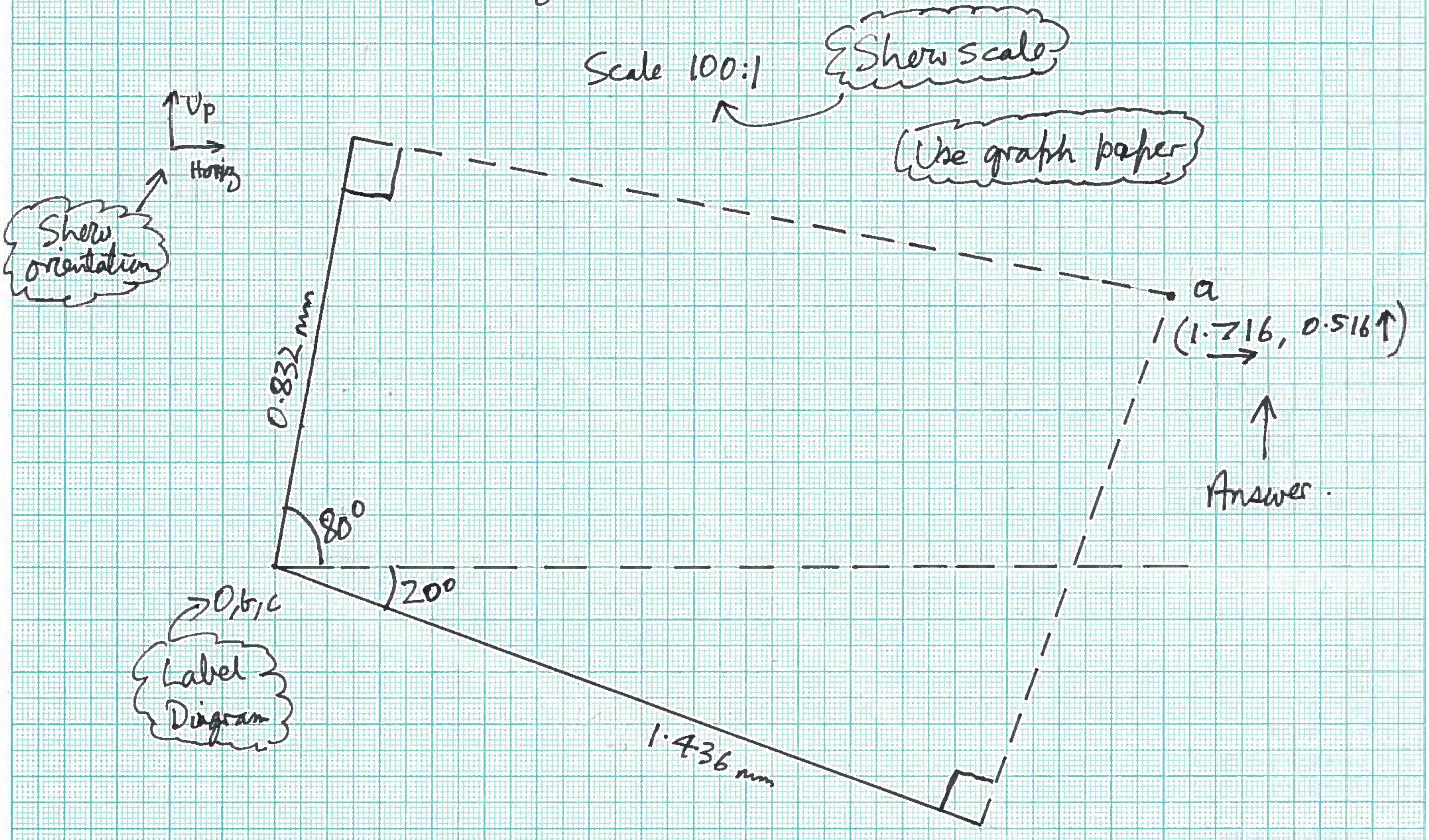
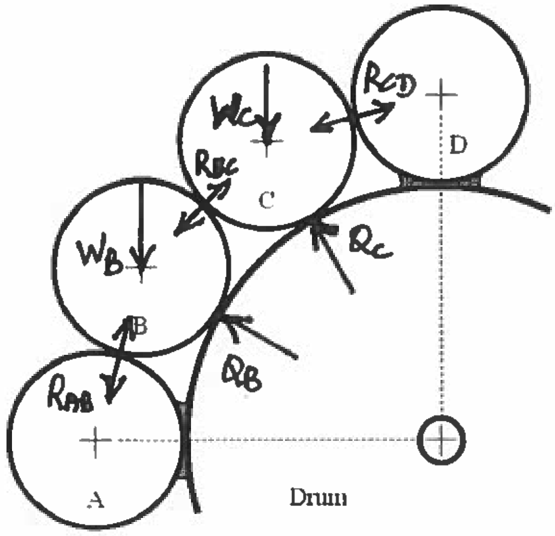
Displacement Diagram.

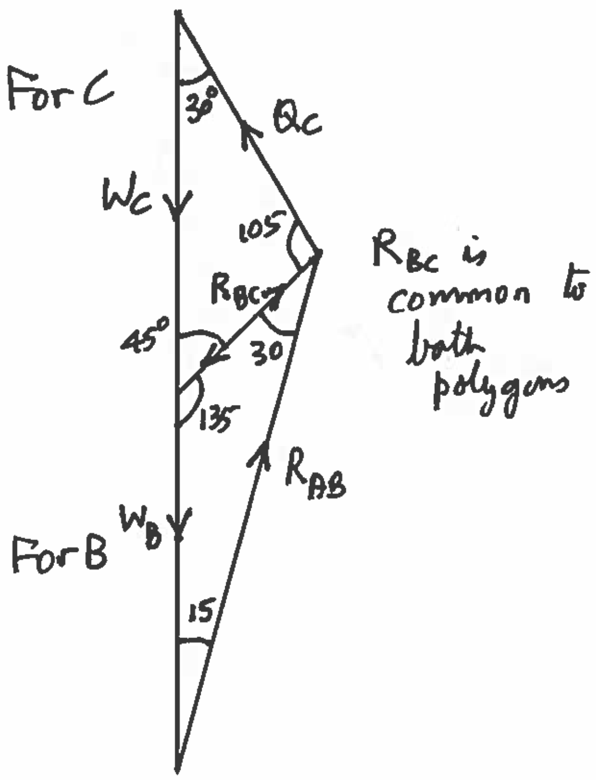


4. *Many candidates couldn't get started on this question because they didn't think clearly.*

First identify all the forces acting



Clearly there are four forces on each of the cylinders B and C, and only three equilibrium equations, so the system is indeterminate. But note that when the cylinders move, B loses contact with the drum, so Q_B=0 and C loses contact with D, so R[redacted]=0. Drawing two polygons of forces leads to the solution. (*Note that many candidates tried to do this problem by writing down equilibrium equations but there are so many different angles involved that very few got the correct answer.*)



By calculation for polygons

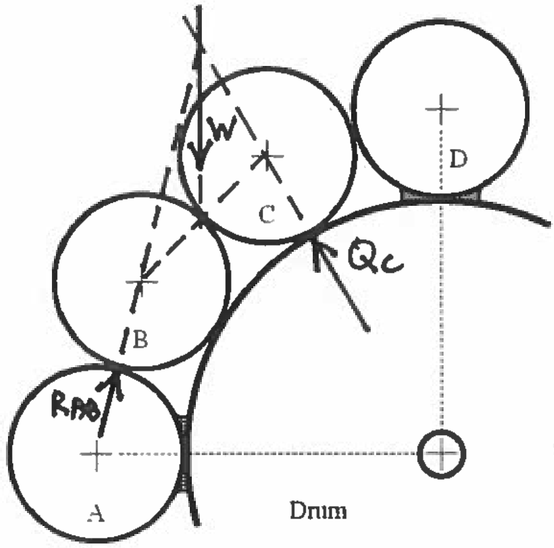
$$\frac{W_C}{\sin 105} = \frac{R_{BC}}{\sin 30}$$

$$\frac{W_B}{\sin 30} = \frac{R_{BC}}{\sin 15}$$

$$R_{BC} = W_C \frac{\sin 30}{\sin 105} = W_B \frac{\sin 15}{\sin 30}$$

$$\Rightarrow W_B = W_C$$

Alternatively, one candidate considered B and C as a single object. This is allowable because even though they slide relative to one another, they remain in contact. It thus becomes a three force problem and simple geometry shows that the resultant force acts through the mid-point of BC, so this must be the centre of gravity of the two cylinders so $W_C = W_B$.

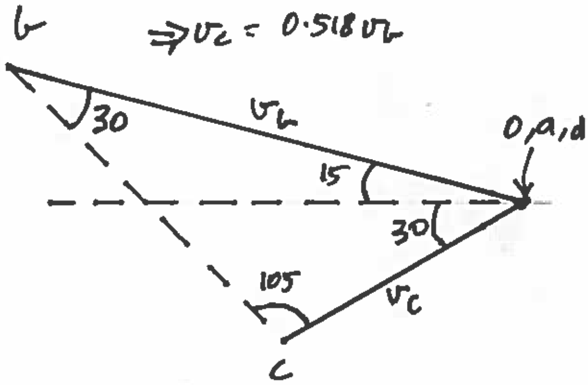


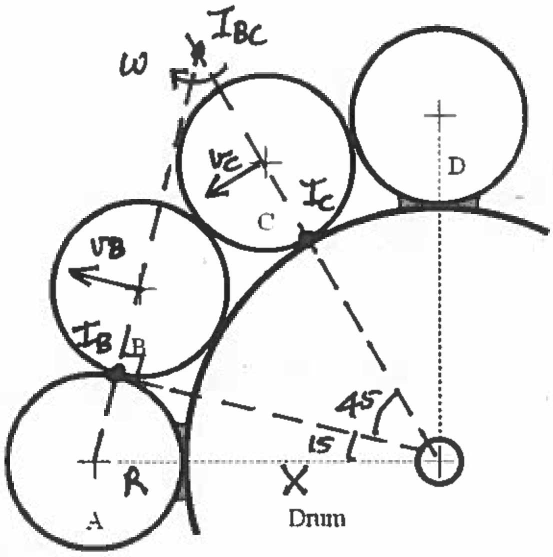
Another alternative is to calculate the speed of motion using a velocity diagram or instantaneous centres and then to calculate the work done against gravity by B and C, which leads to the same result.

Velocity Diagram

$$\frac{v_c}{\sin 30} = \frac{v_b}{\sin 105}$$

$$\Rightarrow v_c = 0.518\, v_b$$





Vertical component

$\uparrow\ v_b \sin 15 = v_b . 0.259$

$\downarrow\ v_c \sin 30 = v_c . 0.259$

Work done $= 0 \quad \therefore W_C = W_B$

By measurement or simple calc

$X = 2.864R\ ;\ OI_B = 3.732R$

$OI_B = 5.278R\ ;\ BI_{BC} = 2.732R$

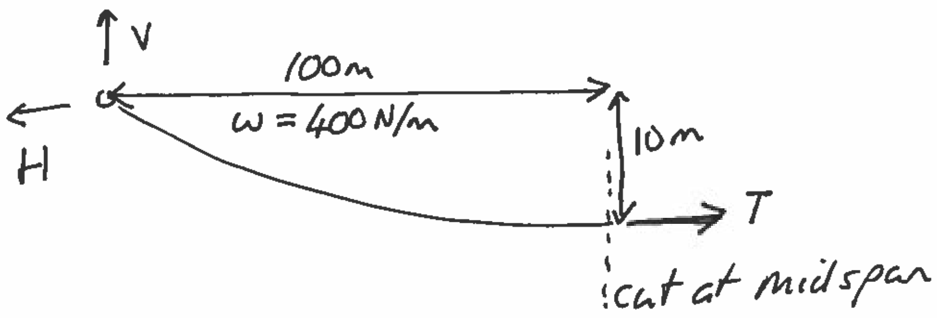
$CI_{BC} = 1.414R$

$\omega = \frac{v_B}{BI_{BC}}$ $\qquad v_c = \omega . CI_{BC} = 0.518\, v_B$

Then as before

5: *This question was generally answered well and many candidates achieved full marks. Several candidates failed to calculate the horizontal load on the cable correctly, as they ignored the self weight of the cable when taking moments about the mid-point. A surprising number of candidates failed to convert the forces on the cable end points to the equal and opposite forces on the truss. A small number of candidates calculated real work instead of virtual work (ie they summed the product of all real bar tensions and extensions), so inadvertently found the total displacement of node C. In this case, treating the horizontal and vertical real loads on the tower separately was unhelpful and led to many algebraic errors, while those candidates who identified that the virtual tensions were only non-zero in two bars saved themselves a lot of time.*

a) Free body diagram for the left half of the cable:



By vertical equilibrium, $V = 400 \times 100 = \underline{40\,kN}$

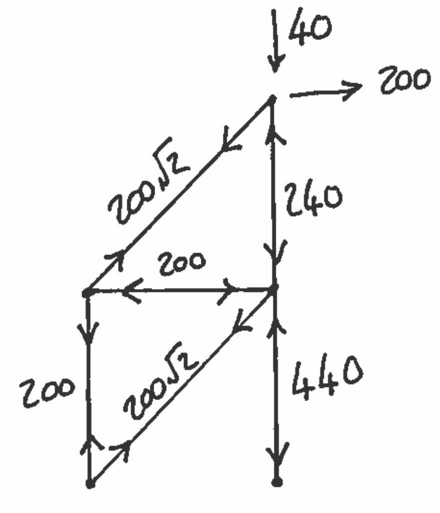
By moments about the cut, $10 \times H + \frac{1}{2} \times 100 \times (100 \times 400) = 100 \times V \therefore \underline{H = 200\,kN}$

These forces act on the cable, so the forces on the tower are equal and opposite: $\downarrow 40\,kN$ and $\rightarrow 200\,kN$.

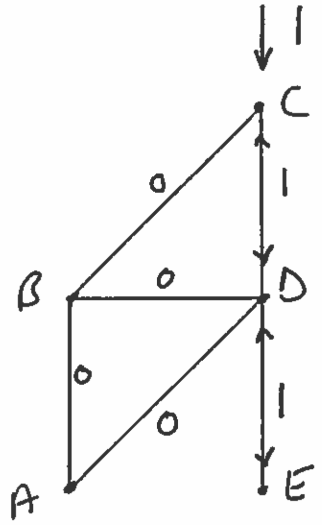
b) The virtual work equation is $\overbrace{\sum F . \underbrace{\delta = \sum T}_{\text{real}} . e}^{\text{virtual}}$. The question asks for a real deflection, so δ and e must be real.

Resolving forces at nodes:

Real:
(units of kN)



Virtual:



Only CD and DE have non-zero vertical tensions, so

$$\sum F^*.\delta = \delta_C^{vert} = \sum T^*.e = e_{CD} + e_{DE} = \frac{L}{AE}.(T_{CD} + T_{DE})$$

$$\therefore \delta_C^{vert} = \frac{10}{0.001 \times 200 \times 10^9} \times (240 \times 10^3 + 440 \times 10^3) = \underline{\underline{34 \text{ mm down}}}$$

c) To avoid yielding, the area of each bar must satisfy $A \geq \frac{|T|}{\sigma_y}$.

Using the real tensions from (b):

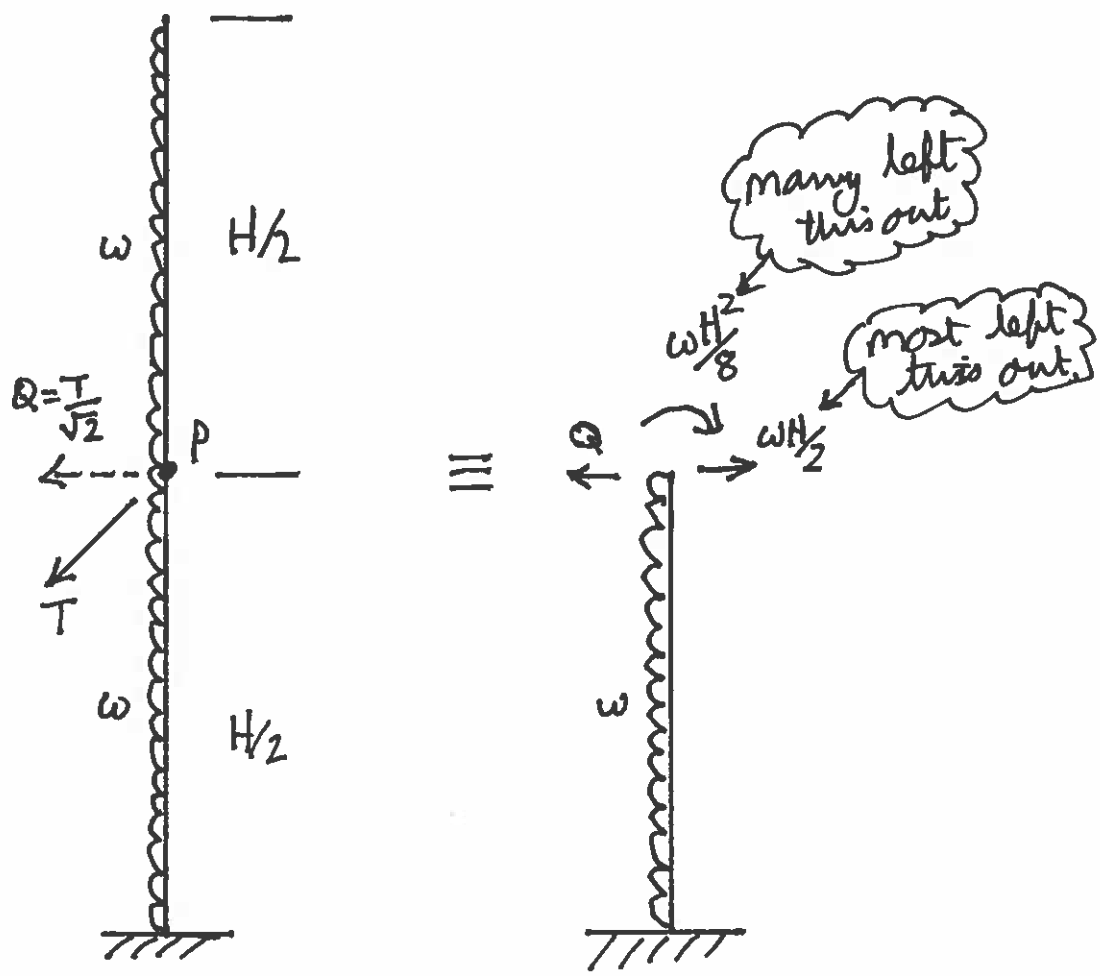
bar	$\lvert T \rvert$ (kN)	$A_{min} = \frac{\lvert T \rvert}{\sigma_y}$ (m^2)	L	$V_{min} = L \times A_{min}$ (m^3)
AB	200	0.0004	10	0.0040
AD	$200\sqrt{2}$	0.00057	$10\sqrt{2}$	0.0080
ED	440	0.00088	10	0.0088
BD	200	0.0004	10	0.0040
BC	$200\sqrt{2}$	0.00057	$10\sqrt{2}$	0.0080
DC	240	0.00048	10	0.0048

$\sum L = 10(4 + 2\sqrt{2})$ $\quad \sum V_{min} = 0.0376\,m^3$

original volume of members $= 0.001 \times \sum L = 0.0683\,m^3$

$\therefore$ potential fraction saving $= 1 - \frac{0.0376}{0.0683} = \underline{\underline{45\%}}$

6(a)

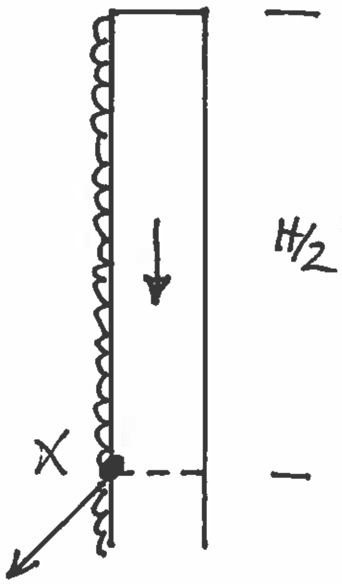


Deflection at P must be zero

$$= \frac{w(H/2)^4}{8EI} + \frac{wH^2/8\,(H/2)^2}{2EI} + \frac{(wH/2 - T/\sqrt{2})\,H/2^3}{3EI} = 0$$

After manipulation $T = \frac{17\sqrt{2}}{16}\,wH = 1.502\,wH$

(b)(i)



Maximum tensile stress at X will be at Point X, on windward side above attachment of tie

(Many candidates calculated the result below the tie)

$$\sigma_{max} = \frac{MR}{I} - \rho g \frac{H}{2}$$

(with $R = 3m$ & thickness $0.1m$ ≈ thin walled

$\therefore I = \pi R^3 t$ or could use $\pi\left(\frac{R^4}{4} - \frac{(R-t)^4}{4}\right)$)

$$M = \frac{wH^2}{8} \quad \therefore \frac{wH^2}{8} \cdot \frac{R}{\pi R^3 t} = \rho_{min}\, g \cdot \frac{H}{2}$$

or $\rho_{min} = \dfrac{wH}{4\pi R^2 g t}$

$H = 100$ m
$w = 2000$ N/m
$R = 3$ m
$t = 0.1$ m

$= \underline{1802 \text{ kg/m}^3}$

Reasonable? Materials data book gives brick ≈ 2000 kg/m³
Concrete ~~2400~~ kg/m³

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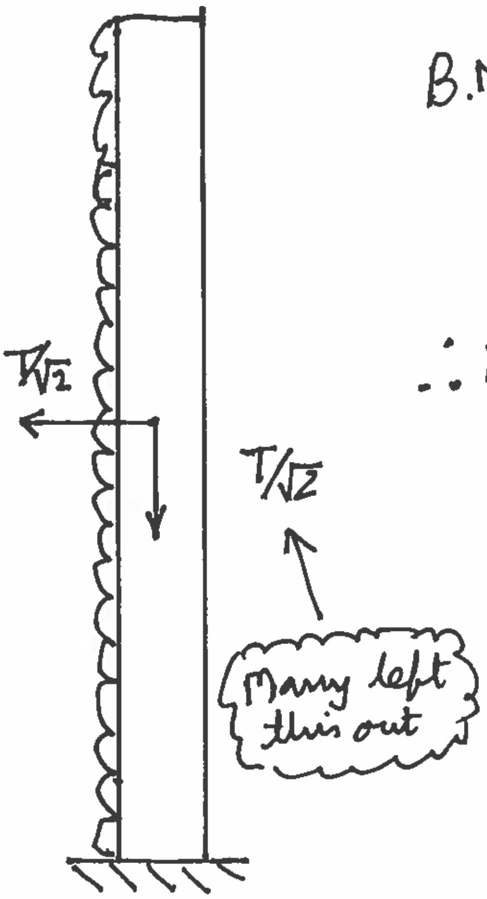
(b)(ii)



B.M at base

$$= \frac{WH^2}{2} - \frac{T}{\sqrt{2}}\cdot\frac{H}{2} = \frac{-wH^2}{32}$$

∴ Moment at base goes down + axial force goes up relative to situation at P

∴ Base will not be critical

Q.E.D.

If you want numbers at base

$$\sigma_{max} = +\frac{WH^2}{32}\cdot\frac{R}{\pi R^3 t} - \rho g H - \frac{T}{\sqrt{2}\cdot 2\pi R t}$$

note difference of sign

$$= \frac{wH^2}{32\pi R^2 t}\left(1 - \frac{8}{\pi} - \frac{17R}{H}\right) = 1.74 \text{ MPa}$$

(compressive)

See overleaf for figures – not part of Question.

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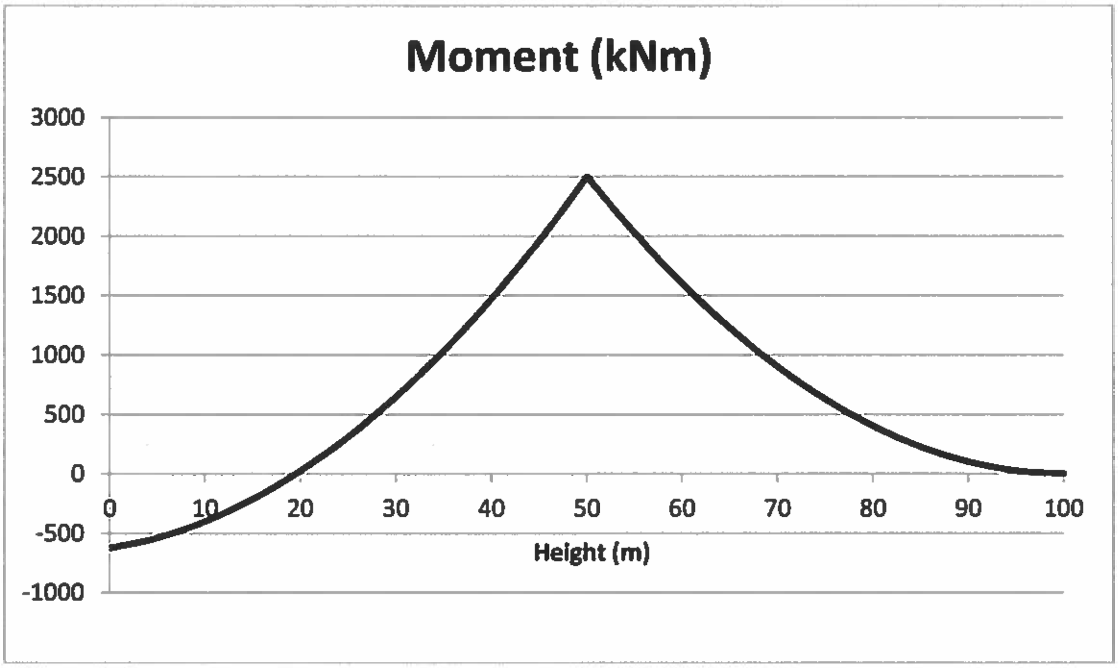
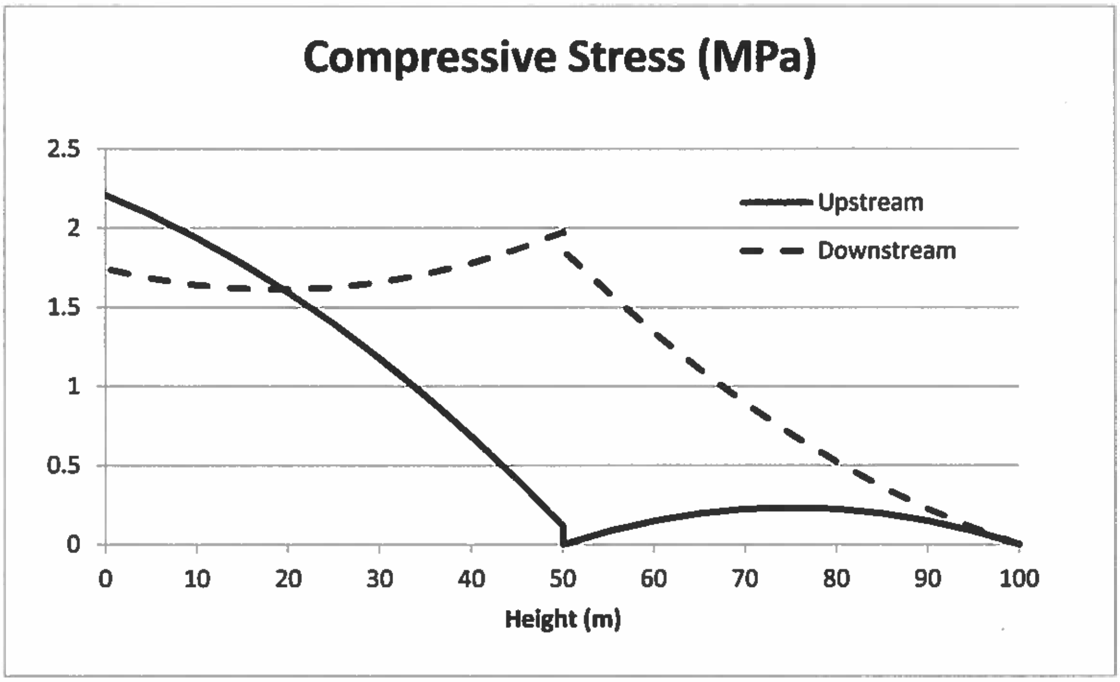
These graphs were not needed for the solution but many candidates clearly assumed that the base of the chimney was critical but these graphs show that this was not the case.

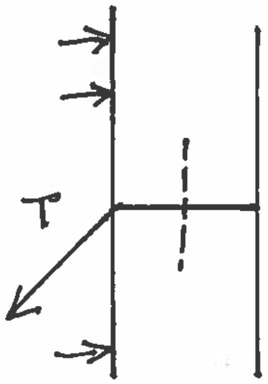
The moment reduces below P (values calculated for critical value density).



The stresses on both faces are all compressive below P



6. Extension



Some good candidates noted that as drawn, T was not attached at centre but at an eccentricity R.

This induces an extra moment $\left(T\frac{R}{\sqrt{2}}\right)$ in part (a) which leads to

$$T = \frac{17\sqrt{2}\ wH}{16\ (1+6R/H)}$$

It has no effect in (b)(i)

Moment at base is now:-

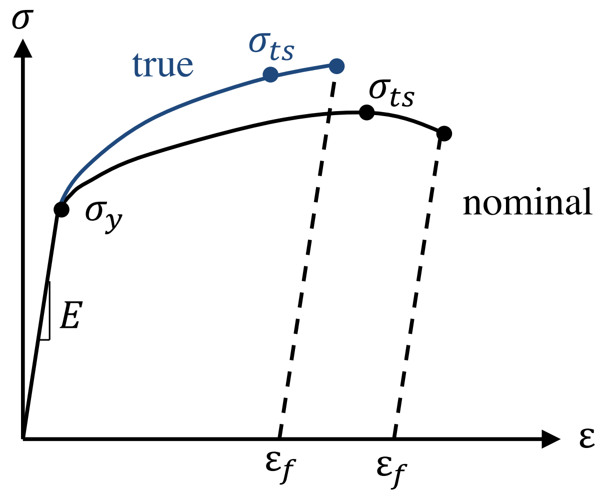
$$\frac{wH^2}{2}\left(1 - \frac{17\,(1+R/H)}{16\,(1+6R/H)}\right)$$

Candidates who noted this effect, even if they chose to ignore it, got bonus marks.

SECTION B

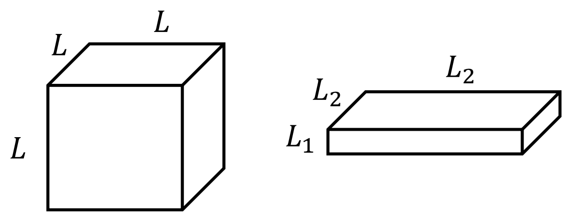
7 (**short**)

(a) Nominal and true tensile stress-strain curves:



[5]

(b) Initial and final dimensions of the cube:



<u>Height of the cube</u>: After one loading to a nominal compressive strain of 0.1, the new height will be $0.9L$. After two it will be $0.9 \times 0.9L$. And so the the final height after compressing n times will be: $L_1 = (0.9)^n L$

Alternatively, if the nominal strain $\varepsilon_n = -0.1$, the true strain for each deformation step is:
$\varepsilon_t = \ln(1+\varepsilon_n) = \ln(0.9)$
True strains are additive, so the total true strain for n steps: $\varepsilon_t = n\ln(0.9)$
Converting this back to nominal strain:
$\varepsilon_n = \exp(n\ln(0.9)) - 1 = (0.9)^n - 1 = -\frac{(L-L_1)}{L} \Rightarrow L_1 = (0.9)^n L$

Width of the cube: Assume conservation of volume during plastic deformation (the material is rigid perfectly plastic, so there is no elastic contribution, though this is often neglected anyway during plastic straining):

$L^3 = L_1 L_2^2$, so $L_2 = \frac{L}{(0.9)^{n/2}}$

[5]

Examiner's comment: For part (a), the nominal stress-strain curve was done well. Most problems occurred with the true case, with few completely correct solutions. Another common problem was forgetting to label key quantities. Part (b) was similar to an examples paper question on a multi-stage 'tandem' rolling mill, but was done surprisingly badly. Some candidates could almost write down the solution for L_1 by inspection, and then went on to use conservation of volume correctly. But a large number over-complicated it. It was common to attempt to solve an elasticity problem, even though the material is rigid-perfectly plastic. Many got in a mess converting between true and nominal strains.

8 (**short**)

(a) Process zone:
○ The process zone is a region of inelastic deformation near to the crack tip.
○ It will locally affect the elastic stress field, and hence the suitability of linear elastic fracture mechanics (i.e. K) to model it.
○ If the process zone is not small compared to the crack length and specimen dimensions (the data-book recommends a factor of 50), $K = K_{IC}$ cannot reliably be used to predict failure. [3]

(b)

- Mg alloy: $\sigma_y = 400$ MPa (Data Book range: $70 - 400$ MPa)
  Fracture toughness (Data Book): $K_{IC} = 12$ to 18 MPa m$^{1/2}$.

  Stress at failure: $\sigma_{0f} = \dfrac{K_{IC}}{3.36\sqrt{\pi a}} = 20.1$ to 30.2 MPa. Higher strength alloys will generally have a lower toughness, so the best estimate would be at the lower end of this range.

  Validity: process zone size (Data Book) $r_p \approx \dfrac{K_{IC}^2}{\pi \sigma_y^2} = 0.286$ to 0.645 mm. This is small compared to the crack length a, and even though it is not quite the recommended 50 times smaller, it is probably reasonable to apply linear elastic fracture mechanics.

- Al alloy: $\sigma_y = 50$ MPa (Data Book range: $30 - 500$ MPa)
  Fracture toughness (Data Book): $K_{IC} = 22$ to 35 MPa m$^{1/2}$.

  Stress at failure: $\sigma_{0f} = \dfrac{K_{IC}}{3.36\sqrt{\pi a}} = 36.9$ to 58.8 MPa. Lower strength alloys will generally have a higher toughness, so the best estimate would be at the upper end of this range.

  Validity: $\sigma_{0f} \sim \sigma_y$, and the process zone size $r_p \approx 62$ to 156 mm $\gg a$. Failure is therefore likely to be by extensive yielding, and so an estimate using linear elastic fracture mechanics is not reasonable. [7]

Examiner's comment: Part (a), a discussion of the process zone size, was done well, with most able to pick up some marks. A common error was to reproduce a sketch from the notes showing a stress-free region around a crack, and discussing strain energy release. A *number of answers were rather unfocussed, fishing for marks. Part (b), plugging numbers into the given stress intensity factor equation, was done surprisingly badly. Most could look up fracture toughness values for the materials given. A mark for picking a value near the appropriate end of the range was claimed by only one or two (most common was to take mid-range values, with the rest assuming high yield strength alloys must have high fracture toughness). It was also surprisingly common to incorrectly take 'a' to be the process zone size, rather than the crack length (given in the question). Discussing the applicability of the stress intensity factor (part (a) was a hint towards the answer) was not done well on the whole. There was a lot of misunderstanding between the physical meaning of the yield stress and the stress at fracture in the presence of crack.*

9 (**short**)

(a) For $t > 0$, $\varepsilon(t) = \varepsilon_0$ is constant, and $\dot{\varepsilon}(t) = 0$. The differential equation becomes

$$0 = \frac{\sigma(t)}{\eta} + \frac{\dot{\sigma}(t)}{E}$$

and is solved by $\sigma(t) = \sigma_0 e^{-Et/\eta}$. At $t = 0^+$, the step of strain implies an infinite strain rate and the dashpot "locks up". Only the spring can respond to the strain, and the stress is then $\sigma(0^+) = \sigma_0 = \varepsilon_0 E$. Finally,

$$\sigma(t) = \varepsilon_0 E \exp\left(-\frac{E}{\eta}t\right) \quad \text{for } t \geq 0\,.$$

as stated. [6]

(b) From the graph, we see that $\sigma(t)$ falls by a factor 2 in 140 s, so that

$$\exp\left(-\frac{E}{\eta}140\right) = \frac{1}{2} \quad \Leftrightarrow \quad \frac{E}{\eta} = \frac{\ln 2}{140} \quad \Leftrightarrow \quad \eta = \frac{0.1\,\mathrm{GPa} \times 140\,\mathrm{s}}{\ln 2}\,,$$

to finally get

$$\eta \approx 20\,\mathrm{GPa\,s}\,.$$

Note that the Maxwell approximation does not fit very well the data, so that answers between about 10 and 50 GPa s may be found, depending on which points are used on the graph. [4]

Examiner's comment: Part (a), solving the differential equation, was done reasonably well using various methods, but the initial condition was rarely justified. For part (b), extracting parameters from an experimental graph, some candidates used the initial slope of the experimental curve, which is not a precise method. Few candidates compared values obtained at different time points, and rightfully concluded the exponential fit was not ideal. Viscosity results given without units, or with the wrong units, were penalised. The units for viscosity could easily be deduced from the time constant of the exponential law given in (a), for example.

10 (**short**)

(a) We use the equations of the Materials Data Book with a volume fraction $V_f = 0.5$ since both materials occupy the same volume in the laminate. Hence:

$$E_{II} = \frac{E_1 + E_2}{2},$$

$$E_{\perp} = 2\left(\frac{1}{E_1} + \frac{1}{E_2}\right)^{-1}. \qquad [3]$$

(b)

- Loading in x direction: loading parallel to the 3 main layers, where the outer layers have $E_{\perp}$ and the inner layer has E_{II}. Hence:

$$E_x = \frac{E_{II} + E_{\perp}}{2} = \frac{1}{2}\left[\frac{E_1 + E_2}{2} + 2\left(\frac{1}{E_1} + \frac{1}{E_2}\right)^{-1}\right].$$

- Loading in y direction: loading parallel to the 3 main layers, where the outer layers have E_{II} and the inner layer has $E_{\perp}$. Hence:

$$E_y = E_x$$

- Loading in z direction: loading perpendicular to the 3 main layers, where both the outer layers and the inner layer have E_{II}. Hence:

$$E_z = 2\left(\frac{1}{E_{II}} + \frac{1}{E_{II}}\right)^{-1} = E_{II} = \frac{E_1 + E_2}{2}. \qquad [7]$$

Examiner's comment: A few candidates obtained a wrong volume fraction, which is a dimensionless quantity, in part (a) about the effective modulus of a laminar composite. For part (b), the vast majority of candidates understood how to obtain the effective moduli of the hybrid composite in relation to part (a). Full marks were only awarded when the calculations were properly explained. The z-direction caused the most problems, even though the same principles as in the other 2 directions apply.

11 (**long**)

(a) Maximum moment (from Fig. 11):

$$M_0 = \frac{F_0 L}{4}$$

Maximum deflection (Structures Data Book, section 4.5.2, case 4):

$$\delta_0 = \frac{M_0 L^2}{6EI}, \quad I = \frac{bd^3}{12}, \quad \therefore\ \delta_0 = \left(\frac{F_0 L}{4}\right)\left(\frac{L^2}{6E}\right)\left(\frac{12}{bd^3}\right) = \frac{F_0 L^3}{2Ebd^3}$$

Maximum stress:

$$\sigma_0 = \frac{M_0 y}{I}, \quad y = \frac{d}{2}, \quad \therefore\ \sigma_0 = \left(\frac{F_0 L}{4}\right)\left(\frac{d}{2}\right)\left(\frac{12}{bd^3}\right) = \frac{3F_0 L}{2bd^2}$$ [6]

(b) (i) Objective: minimise δ_0
Functional constraint: $\sigma_0 \leq \sigma_f$
Geometric constraints: $d =$ free; $L, b =$ fixed [3]

(ii) Functional constraint:

$$\sigma_0 = \sigma_f = \frac{3F_0 L}{2bd^2} \quad \Rightarrow \quad d = \sqrt{\frac{3F_0 L}{2b\sigma_f}}$$

Substitute free variable into the objective equation:

$$\delta_0 = \frac{F_0 L^3}{2Eb}\left(\frac{2b\sigma_f}{3F_0 L}\right)^{3/2} = \frac{\sigma_f^{3/2}}{E}\left[\frac{1}{2}\left(\frac{2L}{3}\right)^{3/2}\left(\frac{b}{F_0}\right)^{1/2}\right]$$

Therefore maximise the material index: $E/\sigma_f^{3/2}$ [5]

(iii) Using the Young's modulus - strength chart, and the line corresponding to $\sigma_f^{3/2}/E = C$, the best options will be in the top-left corner of the chart where E is high and σ_f low. The best choices are therefore: Cu alloys, Pb alloys and Al alloys.

Natural materials and composites are not suitable, as it is not possible to manufacture components on this scale. The device has dimensions $\sim$ µm, which is likely to be smaller than the microstructural length scales of these materials (e.g. the cell size in natural materials, or the size of reinforcing particles or fibres in composites). [4]

(c) (i)

- Constraint 1: maximum deflection

$$\delta_0 = \frac{F_0 L^3}{2Ebd^3} \leq \delta_{max} = 2\,\mu\text{m}, \quad \therefore d \geq \left(\frac{F_0 L^3}{2Eb\delta_{max}}\right)^{1/3}$$

- Constraint 2: maximum stress

$$\sigma_0 = \frac{3F_0 L}{2bd^2} \leq \sigma_f, \quad \therefore d \geq \left(\frac{3F_0 L}{2b\sigma_f}\right)^{1/3}$$

Evaluate d for both constraints:

Material	Constraint 1, d(μm)	Constraint 2, d(μm)
Silicon nitride	<u>5.57</u>	3.16
Aluminium	8.94	<u>12.2</u>
Nickel	6.30	<u>7.07</u>

The value of d for the active constraint (the largest value for each material) is underlined. [8]

(ii) For each value of d, evaluate the objective: $m = \rho Lbd$.

Material	Mass m (kg $\times 10^{-12}$)	Rank
Silicon nitride	6.68	1
Aluminium	13.2	2
Nickel	25.2	3

[4]

Examiner's comment: This question was done well by most. Most candidates were able to derive the formulae correctly in part (a). A number chose not to use the recommended section of the data book though, opting instead to superimpose cantilever cases. Part (b) is a conventional performance index derivation, and most knew the correct procedure. Most marks were lost applying the material index to the data book selection chart, and shortlisting materials. Common errors were getting the slope of the line wrong, or heading to the wrong corner of the chart to maximise the index. Generic remarks on the unsuitability of composites and natural materials, not considering the micron-scale of the part, didn't score well. Part (c) increased the complexity by changing the objective and adding two constraints. Again, most knew the correct procedure for identifying the active constraint and then minimising the objective. Most marks were lost with errors plugging numbers in to evaluate the free variable ('d'). This was complicated by the dimensions being in microns, which seems to have caused calculator problems.

12 (**long**)

(a) (i) The mass per unit x-length of the upper portion of the wall between the distance z and H from the base is given by $\rho A(z)$, where $A(z)$ is the area of the portion in the (y,z)-plane:

$$\begin{aligned}
A(z) &= \int_z^H D(z')\mathrm{d}z' \\
&= \int_z^H D_0 \exp\left(-\frac{\rho g}{P}z'\right)\mathrm{d}z' = -D_0\frac{P}{\rho g}\left[\exp\left(-\frac{\rho g}{P}z'\right)\right]_z^H \\
&= D_0\frac{P}{\rho g}\left[\exp\left(-\frac{\rho g}{P}z\right) - \exp\left(-\frac{\rho g}{P}H\right)\right].
\end{aligned}$$

Finally the mass per unit length of the upper portion is

$$\rho A(z) = \frac{PD_0}{g}\left[\exp\left(-\frac{\rho g}{P}z\right) - \exp\left(-\frac{\rho g}{P}H\right)\right] = \frac{P}{g}\left[D(z) - D(H)\right]. \qquad [5]$$

(ii) Consider the free body diagram of the section of the wall between its base and the distance z from the ground. The vertical forces per unit x-length acting on the section of width $D(z)$ are the pressure $PD(H)$ and the weight of the upper portion $\rho gA(z)$. From the previous question we see that the total force F_z per unit length is

$$F_z(z) = PD(H) + \rho gA(z) = PD(z).$$

Dividing $F_z(z)$ by $D(z)$ to get the stress, we finally obtain:

$$\sigma_z(z) = P \quad \text{independent of } z. \qquad [3]$$

(iii) 3D Hooke's law in the wall reads:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z), \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z), \\
\varepsilon_z &= \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y).
\end{aligned}$$

With $\varepsilon_x = 0$ (infinitely long in the x-direction), $\sigma_y = 0$ (slender wall) and $\sigma_z = P$ (from the previous question), the first equation gives $\sigma_x = \nu P$ and the last one finally gives

$$\varepsilon_z = \frac{1}{E}(P - \nu\sigma_x) = \frac{(1-\nu^2)P}{E}. \qquad [7]$$

(iv) For low strains, $\varepsilon_z = \dfrac{\Delta H}{H}$, so that:

$$\Delta H = \frac{(1-\nu^2)HP}{E}.$$

With the conventions used here (positive compressive strain), $\Delta H > 0$ corresponds to a decrease of the height from the undeformed state. Using the values given for concrete,

$$\frac{\Delta H}{H} = \frac{(1-0.2^2)\times 0.8\times 20\,\mathrm{MPa}}{17\times 10^3\,\mathrm{MPa}} \approx 9\times 10^{-4} \approx 0.1\,\%.$$

This is indeed a small strain. [3]

(b) (i) From Hooke's law in 3D, the strain's x,y-components in the adhesive read:

$$\varepsilon_{\mathrm{a},x} = \frac{1}{E_\mathrm{a}}(\sigma_{\mathrm{a},x} - \nu_\mathrm{a}\sigma_{\mathrm{a},y} - \nu_\mathrm{a}\sigma_{\mathrm{a},z}),$$
$$\varepsilon_{\mathrm{a},y} = \frac{1}{E_\mathrm{a}}(\sigma_{\mathrm{a},y} - \nu_\mathrm{a}\sigma_{\mathrm{a},x} - \nu_\mathrm{a}\sigma_{\mathrm{a},z}).$$

Here $\varepsilon_{\mathrm{a},x} = \varepsilon_{\mathrm{a},y} = 0$ (the adhesive remains bonded) and $\sigma_{\mathrm{a},z} = P$. Inspection of the two relations above indicates $\sigma_{\mathrm{a},x} = \sigma_{\mathrm{a},y}$. Substituting in one of them gives:

$$\sigma_{\mathrm{a},x} = \sigma_{\mathrm{a},y} = \frac{\nu_\mathrm{a}P}{1-\nu_\mathrm{a}}.$$ [7]

(ii) Since the thermal strain is compressive, it must be added as a positive strain in each direction (convention of positive compressive strain). Hence the Hooke's law in the x and y directions now reads:

$$\varepsilon_{\mathrm{a},x} = \frac{1}{E_\mathrm{a}}(\sigma_{\mathrm{a},x} - \nu_\mathrm{a}\sigma_{\mathrm{a},y} - \nu_\mathrm{a}\sigma_{\mathrm{a},z}) + \alpha|\Delta T|,$$
$$\varepsilon_{\mathrm{a},y} = \frac{1}{E_\mathrm{a}}(\sigma_{\mathrm{a},y} - \nu_\mathrm{a}\sigma_{\mathrm{a},x} - \nu_\mathrm{a}\sigma_{\mathrm{a},z}) + \alpha|\Delta T|.$$

The conditions $\varepsilon_{\mathrm{a},x} = \varepsilon_{\mathrm{a},y} = 0$ (the adhesive remains bonded) and $\sigma_z = P$ are not changed. Again, inspection of these two relations above indicates $\sigma_{\mathrm{a},x} = \sigma_{\mathrm{a},y}$. Substituting into one of them now gives:

$$\sigma_{\mathrm{a},x} = \sigma_{\mathrm{a},y} = \frac{\nu_\mathrm{a}P - E_\mathrm{a}\alpha|\Delta T|}{1-\nu_\mathrm{a}},$$

indicating that the compressive horizontal stress is reduced upon cooling. [5]

Examiner's comment: This was the final question on the paper, and candidates appear to run out of time. Part (a) considered a wall whose geometry is such that its stress under self-weight is uniform throughout. The algebra was simpler than in similar examples paper questions. But, a lot of candidates took very long routes to solve the system of equations from Hooke's law, increasing the likelihood of mistakes along the way, and ended up running out of time. The numerical result was rarely commented. In part (b), the calculations based on 3D Hooke's law were again very similar to the ones found in the examples paper, but a large number of candidates wasted a lot of time deriving the result $\sigma_{a,x} = \sigma_{a,y}$ *that can be readily justified with symmetry. The last part dealt with thermal strain, which seems to be well understood, and was generally addressed correctly by the candidates who had the time to attempt it. Among them, most correctly used their physical intuition to explain the change in stress with temperature.*

END OF PAPER