EGT0: ENGINEERING TRIPOS PART IA

**Paper 2: STRUCTURES AND MATERIALS**

Solutions

**SECTION A**

1 The bending moment distribution is symmetric about $x=0$, so for $x \geq 0$, the moments are,

$$M(x) = M_0 \frac{L-x}{L}$$

The deflection (downwards) of the beam, $v$ satisfies,

$$M(x) = -EI\frac{d^2v}{dx^2}$$

subject to the boundary conditions at $x=0$ $\frac{dv}{dx}=0$ (symmetric loading of a symmetric beam) and at $x=L$ $v=0$ (the end is on a pin jointed support)

Substituting the bending moment distribution into the differential equation, integrating twice and applying the relevant boundary condition to find the integration constant each time, we get:

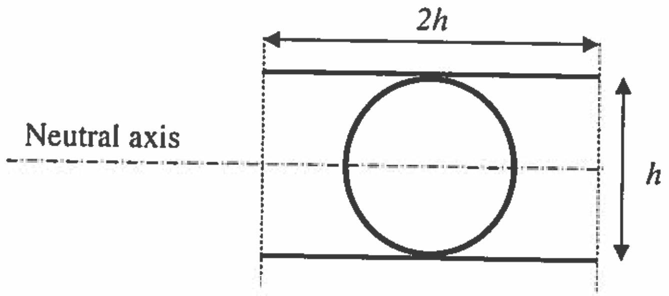
$$M_0\frac{(L-x)}{L} = -EI\frac{d^2v}{dx^2}$$

$$M_0\frac{2Lx-x^2}{2L} = -EI\frac{dv}{dx}$$

$$M_0\frac{3Lx^2-x^3-2L^3}{6L} = -EIv$$

Therefore: $v(x) = \frac{M_0}{EI}\frac{x^3-3Lx^2+2L^3}{6L}$

2 The horizontal distribution of material (parallel to the neutral axis) does not affect the second moment of area of the beam, so the material within the distance $\lambda$ (which must equal *2h*) can be rearranged to look like:



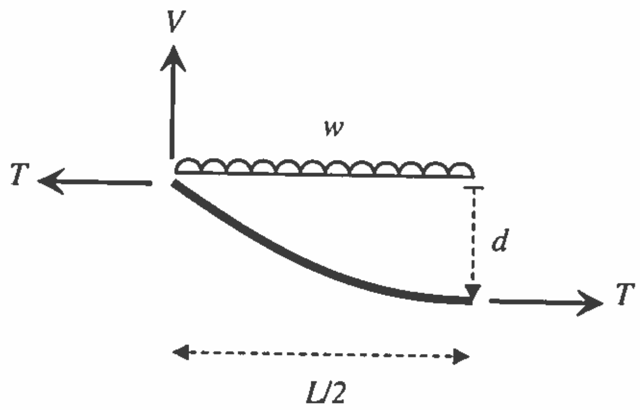

This now looks like a thin-walled tube with two rectangular flanges, so using the data book formula for the thin walled tube, and the parallel axis theorem for the two flanges (ignoring their second moment of area about their own neutral axes, as instructed), we obtain for one wavelength of the material

$$I_{wavelength} = \pi\left(\frac{h}{2}\right)^3 t + 2\times(2h\times t)\times\left(\frac{h}{2}\right)^2 = \frac{h^3 t}{8}(\pi+8)$$

Averaging over $\lambda$, gives,

$$I_{per\ unit\ width} = \frac{h^2 t}{16}(\pi+8)$$

3 The figure below is a free body diagram of half the span of a cable between two floats with the net download load described as $w$. By vertical equilibrium, $V = wL/2$ (although the total vertical force provided by each float must be twice this as the float supports the cable to either side.)




Taking moments about the right end of the cable in this free-body diagram,

$$Td + \frac{w}{2}\left(\frac{L}{2}\right)^2 = \frac{wL}{2} \times \frac{L}{2}$$

$$\therefore d = \frac{wL^2}{8T}$$

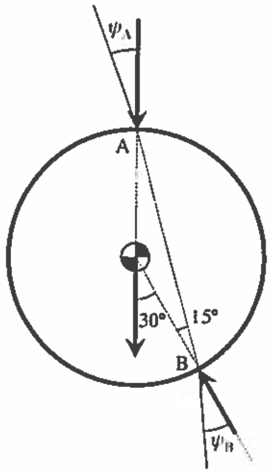
The net downward load $w$ is the difference between the weight per unit length of the cable itself and the buoyancy created by the displaced water, of density $\rho_w$. Therefore,

$$w = (\rho - \rho_w) gA$$

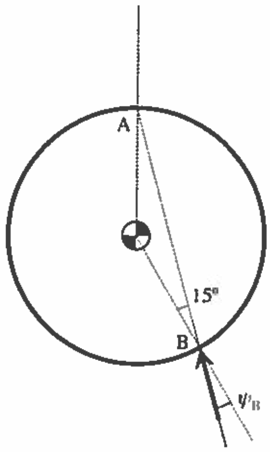
So the mid-span dip is related to the material density by,

$$d = \frac{(\rho - \rho_w) gAL^2}{8T}$$

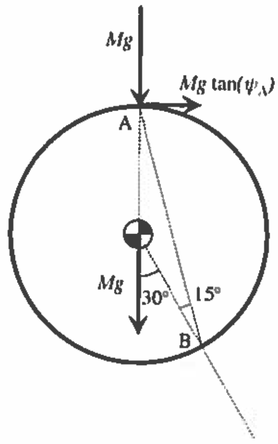
4 The three figures below show how the limiting values of the two angles of friction are found




Free body diagram (reaction forces shown for frictionless contact

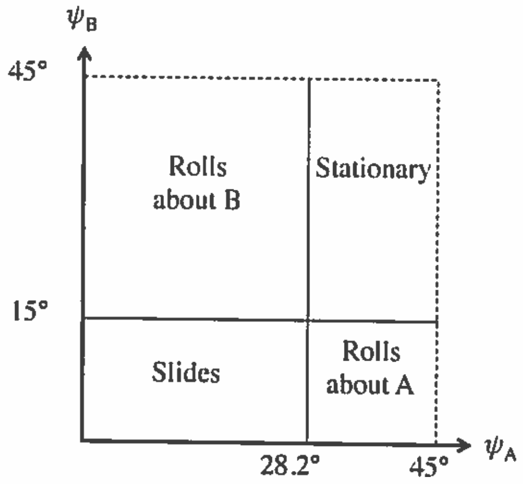



Moments about A: the cylinder can only be in equilibrium if $\psi_B \geq 15°$. Otherwise it will slip at B.

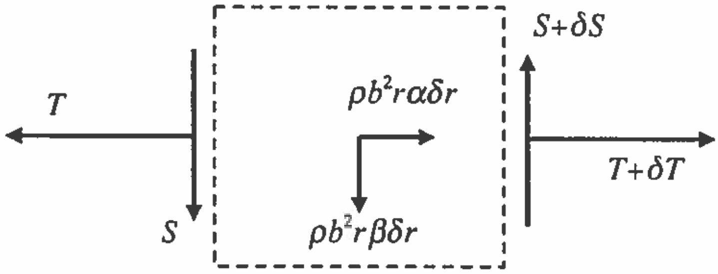



Moments about B: the cylinder can only be in equilibrium if $2MgR\sin 30 \leq Mg\tan\psi_A R(1+\cos 30)$ *i.e.* $\psi_A \geq 28.2°$. Otherwise it will slip at A.

The resulting graph of the four possible instantaneous responses for the cylinder is:



5 (a) Expanding the view of the infinitessimal section in the question as a free body diagram, we get




The bar is therefore subject to a distributed load, $w(r)$ perpendicular to its length, so $\frac{d^2M}{dr^2} = w(r)$. At $r = L$ the Shear force and moment are both zero, so integrating twice and applying these boundary conditions we get,

$$S(r) = \tfrac{1}{2}\rho b^2 \beta (r^2 - L^2)$$
$$M(r) = \tfrac{1}{6}\rho b^2 \beta (r^3 - 3L^2 r + 2L^3)$$

(b) From horizontal equilibrium of the above figure, the axial force $T$ which must be zero at $r = L$, satisfies

$$\frac{dT}{dr} + \rho b^2 r\alpha = 0 \quad and\ T(L) = 0$$
$$\therefore T = \tfrac{1}{2}\rho b^2 \alpha (L^2 - r^2)$$

The greatest longitudinal stress in the bar is on the edge where bending leads to tension and comprises the bending stress plus the stress due to the axial force $T$, which is uniform across the cross-section. Thus,

$$\sigma_{\max}(r) = \frac{My}{I} + \frac{T}{A}$$
$$= \frac{\rho\beta}{b}(r^3 - 3L^2 r + 2L^3) + \frac{\rho\alpha}{2}(L^2 - r^2)$$

The greatest shear stress is at the neutral axis of the bar, and by the standard formula is,

$$\tau_{max}(r) = \frac{S(r) A_c \bar{y}}{bI}$$
$$= \frac{3\rho\beta}{4}(r^2 - L^2)$$

(c) The two limiting criteria specified are:

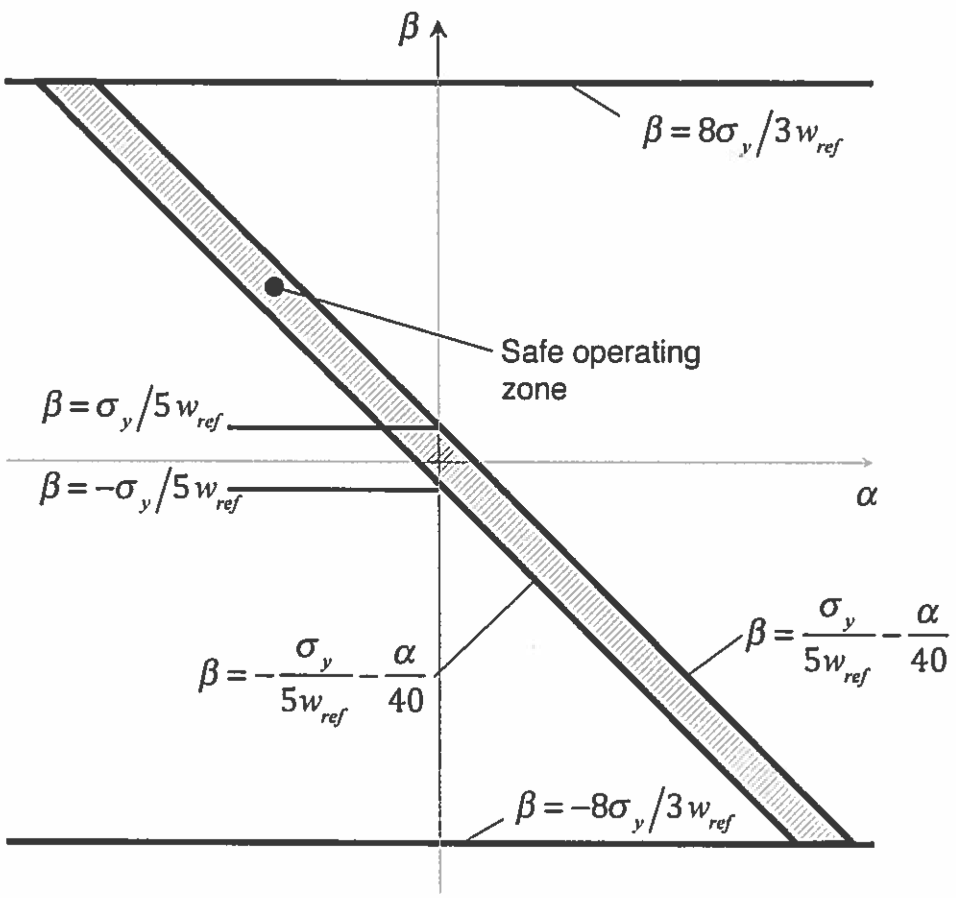
$$|\sigma_{max}(0)| \le \sigma_y \qquad \qquad |\tau_{max}(0)| \le \tfrac{1}{2}\sigma_y$$

$$\therefore \left| \frac{2L^3\rho\beta}{b} + \frac{\rho\alpha L^2}{2} \right| \le \sigma_y \quad \text{and} \quad \therefore \left| \frac{-3\rho\beta L^2}{2} \right| \le \sigma_y$$

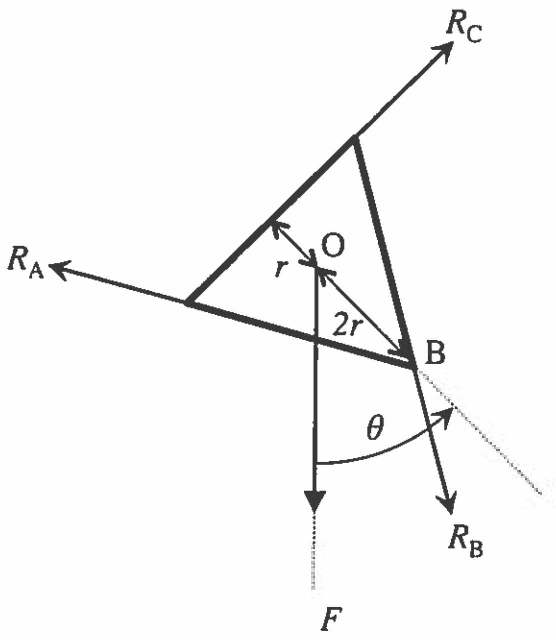
Applying the values given in the question, the four limiting lines defined by these two inequalities are:

$$\beta = \frac{\sigma_y}{5w_{ref}} - \frac{\alpha}{40}, \quad \beta = -\frac{\sigma_y}{5w_{ref}} - \frac{\alpha}{40}, \quad \beta = \frac{8\sigma_y}{3w_{ref}}, \quad \beta = \frac{-8\sigma_y}{3w_{ref}}$$

This therefore leads to the plot of safe use of the bar as:



6 (a) Draw a free body diagram for the forces acting on the triangular hub:




*(O is at the centroid of the triangle, hence the two distances* r *and* 2r *are known.)*

Take moments about the point B to eliminate $R_A$ and $R_B$, equilibrium is established when,

$$3r \times R_C = 2r \sin\theta \times F$$

Hence,

$$R_C = \tfrac{2}{3} F \sin\theta$$

(b) The wheel has rotational symmetry order three, so the forces in the other two spokes must be the same as those in spoke Cc, rotated appropriately, so,

$$R_A = \tfrac{2}{3} F \sin(\theta + 2\pi/3)$$
$$R_B = \tfrac{2}{3} F \sin(\theta + 4\pi/3)$$

By standard formulae, the extension $e$ in a spoke with the properties specified in the question subject to a force $R$ is, $e = LR/AE$. We can now equate the external and internal work (both are real, not virtual) done as the force $F$ is applied to the wheel, and the centre of the hub O deflects downwards a distance $d(\theta)$:



(TURN OVER

*External work = Internal work*

$$F \cdot d(\theta) = R_A \cdot e_A + R_B \cdot e_B + R_C \cdot e_C$$

$$= \tfrac{4}{9}\frac{LF^2}{AE}\left[\sin^2\theta + \sin^2(\theta + 2\pi/3) + \sin^2(\theta + 4\pi/3)\right]$$

$$d(\theta) = \tfrac{4}{9}\frac{LF}{AE}\left[\sin^2\theta + \sin^2(\theta + 2\pi/3) + \sin^2(\theta + 4\pi/3)\right]$$

This could be a satisfactory answer to the question, although it is difficult to determine the dependency of $d$ on the angle of rotation in this case. However, expanding the term in square brackets using the double angle formulae,

$$\sin^2\theta + \sin^2(\theta + 2\pi/3) + \sin^2(\theta + 4\pi/3)$$

$$= \sin^2\theta + \left(\tfrac{-1}{2}\sin\theta + \tfrac{\sqrt{3}}{2}\cos\theta\right)^2 + \left(\tfrac{-1}{2}\sin\theta - \tfrac{\sqrt{3}}{2}\cos\theta\right)^2$$

$$= \sin^2\theta + \left(\tfrac{1}{2}\sin^2\theta + \tfrac{3}{2}\cos^2\theta\right)$$

$$= \frac{3}{2}$$

Therefore, for this particular design of wheel, the downwards displacement of the hub is constant regardless of the rotation of the wheel, and is,

$$d = \tfrac{2}{3}\frac{LF}{AE}$$

A sketch of this is:

## SECTION B

**7 (short)**

(a) The three main hardening mechanisms are:

(i) *Work hardening*: Gliding dislocations on different slip planes interact, and pinning occurs. The gliding dislocation bows out, requiring an increase in shear stress, until the pinning point gives way.
*Key microstructural parameters*: Dislocation density, and hence spacing between the pinning points. [2]

(ii) *Solid solution hardening*: Solute atoms act as obstacles to gliding dislocations. Dislocations bow out until the line tension pulls the dislocation past the solute atom.
*Key microstructural parameters*: Solute concentration, and hence spacing between solute atoms. [2]

(iii) *Precipitation hardening*: Alloying elements form compounds, distributed as small particles (precipitates) within the lattice, providing pinning points for dislocations.
*Key microstructural parameters*: Size and volume fraction of precipitates, and hence the spacing between them. [2]

(b) Working in units of MPa and mm, substitute the values into the given equation:

$$33 = \sigma_0 + \frac{k}{\sqrt{0.5}}\ , \ 38 = \sigma_0 + \frac{k}{\sqrt{0.1}}$$

Eliminating $\sigma_0$ gives: $k = \frac{38-33}{\frac{1}{\sqrt{0.1}} - \frac{1}{\sqrt{0.5}}} = 2.86\ \text{MPa}\sqrt{\text{mm}} = 0.0904\ \text{MPa}\sqrt{\text{m}}$

And therefore: $\sigma_0 = 33 - \frac{k}{\sqrt{0.5}} = 29.0$ MPa.

Hence, for $\sigma_y = 45$ MPa, the grain size: $d = \left(\frac{k}{\sigma_y - \sigma_0}\right)^2 = 0.0318$ mm. This is a relatively weak hardening mechanism for polycrystalline metals of typical grain sizes, as the much closer obstacle spacing present in the mechanisms above provides more effective resistance to dislocation motion. [4]

8 **(short)**

(a) The FCC unit cell is shown in Fig. 1. If the unit cube has side length $d$, the length of the diagonal on a face is:

$$\sqrt{2}d = 4R\,, \quad \therefore d = 2\sqrt{2}R$$

The volume of the unit cell: $V = d^3 = 16\sqrt{2}R^3$ [4]

(b) The number of atoms in a unit cell:

$$n = \text{corner atoms} + \text{face atoms} = 8 \times \frac{1}{8} + 6 \times \frac{1}{2} = 4$$

If the atomic weight $M = 63.5$ kg kmol$^{-1}$, and Avogadro's number $N_A = 6.022 \times 10^{26}$ kmol$^{-1}$, the mass of the unit cell:

$$m = \frac{nM}{N_A} = \frac{4 \times 63.5}{6.022 \times 10^{26}} = 421.8 \times 10^{-27}\ \text{kg}$$

The density is therefore:

$$\rho = \frac{m}{V} = \frac{421.8 \times 10^{-27}}{16\sqrt{2} \times (0.128 \times 10^{-9})^3} = 8888\ \text{kg m}^{-3}$$

The data book value is $\rho = 8930$kg m$^{-3}$, so the calculation is within 0.5% of the measured value. Differences may be due to defects in the crystal structure (e.g. at grain boundaries), and the presence of impurities or alloying elements. [6]

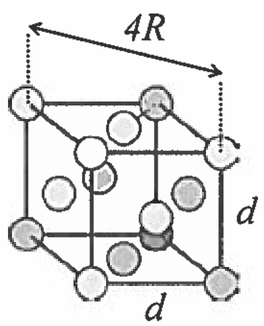



Fig. 1

9 **(short)**

(a) The volume of 25 kg of steel parts (noting that that there may be some variation in the assumed density of steel):

$$V = \frac{m_{\text{steel}}}{\rho_{\text{steel}}} = \frac{25\ \text{kg}}{7850\ \text{kg m}^{-3}} = 3.2 \times 10^{-3}\ \text{m}^3$$

The mass of the Al alloy parts, if the dimensions (and therefore volume) are the same:

$$m_{\text{Al}} = V\rho_{\text{Al}} = 3.2 \times 10^{-3} \times 2700 = 8.64\ \text{kg}$$

The mass saving is: $\Delta m = m_{\text{steel}} - m_{\text{Al}} = 25 - 8.64 = 16.4$ kg .

The change in embodied $CO_2$ is:

$\Delta CO_2 = CO_{2(\text{Al})} - CO_{2(\text{steel})} = (8.64 \times 8.24) - (25 \times 1.37) = 71.2 - 34.3 = +37.0\ \text{kg}\ CO_2$

Therefore, the weight is reduced by 34%, but the embodied $CO_2$ is increased by 108%. [4]

(b) The reduction in $CO_2$ emissions per km travelled, due to the weight reduction:

$$\left(\frac{1 \times 10^{-3}}{10}\right) \times 16.4\ \text{kg} = 1.64 \times 10^{-3}\ \text{kg}\ CO_2\ \text{km}^{-1}$$

The reduction in emissions over 1 year (10 000 km) is:

$$\Delta CO_2 = 1.64 \times 10^{-3} \times 10\,000 = 16.4\ \text{kg}\ CO_2$$

To recover the additional embodied $CO_2$ therefore requires: $\frac{37.0}{16.4} = 2.3$ years.

This is less than the typical lifetime of a vehicle, so there is likely to be a net benefit from the change in material. Recycling the Al alloy at the end of life would further increase the benefit. Note that load bearing steel parts may not be replaceable by Al alloy parts of the same dimensions, due to differences in mechanical properties (e.g. Al alloy has a lower Young's modulus). Also, this calculation does not consider the material costs. [6]

10 (**short**)

(a) The total strain in the strip (elastic + thermal) is constrained to be zero. Taking tensile strains and temperature increases to be positive:

$$\varepsilon = \varepsilon_{el} + \varepsilon_{th} = \frac{\sigma}{E} + \alpha\Delta T = 0\,, \ \therefore \Delta T = -\frac{\sigma}{E\alpha}$$

If the strip yields in tension, $\sigma = +\sigma_y$, and so the temperature change:

$$\Delta T = \frac{-200 \times 10^6}{\left(70 \times 10^9\right)\left(24 \times 10^{-6}\right)} = -119\ \text{K}$$

i.e. the strip is cooled by 119K. [4]

(b) Because the large block doesn't expand thermally, and can be assumed stress free (so no elastic strains), its total strain is zero. Because the thin Al alloy layer is bonded to it, its total strain is therefore also constrained to be zero. However, the Al alloy layer will now be in a state of bi-axial stress, and so:

$$\varepsilon_{el} = \frac{\sigma}{E} - \nu\frac{\sigma}{E} = \left(\frac{1-\nu}{E}\right)\sigma$$

Hence, again taking tensile strains and temperature increases to be positive:

$$\varepsilon = \varepsilon_{el} + \varepsilon_{th} = \left(\frac{1-\nu}{E}\right)\sigma + \alpha\Delta T = 0\,, \ \therefore \sigma = -\left(\frac{E}{1-\nu}\right)\alpha\Delta T$$

For a temperature rise $\Delta T = +5$ K:

$$\sigma = -\left(\frac{70 \times 10^9}{1-0.3}\right)\left(24 \times 10^{-6} \times 5\right) = -12\ \text{MPa}$$

The Al alloy layer is therefore in compression. [6]

**11 (long)**

(a) (i) When the surfaces are pressed together, at the microscopic level contact will be through surface asperities. The softer steel A will be indented by the asperities of steel B. At each contact, the force transmitted $F_c \approx H_A a_c$, where $a_c$ is the local area of contact. The total force will be the sum of the contributions from each asperity:

$$F = \sum F_c \approx \sum (H_A a_c) = H_A \sum a_c = H_A A_t$$

The true area of contact is therefore: $At \approx F/H_A$ . [4]

(ii) The wear rate:

$$Q = \frac{KF}{H_A} = \frac{\text{vol. removed}}{\text{sliding dist.}}$$

For a short time $\delta t$:

$$\text{vol. removed} = (2\pi Rh)\,\delta L = (2\pi Rh)\,\frac{dL}{dt}\delta t \ , \quad \text{sliding dist.} = \omega R \delta t$$

Substituting into the wear rate and rearranging gives the rate of shortening: [4]

$$\frac{dL}{dt} = \frac{Q\omega}{2\pi h} = \frac{KF}{H_A}\frac{\omega}{2\pi h}$$

(b) (i) Fig. 2 shows the ranges of $N_f$ corresponding to low cycle fatigue and high cycle fatigue. The stress range at the endurance limit: $\Delta\sigma_e = 270$ MPa. [6]

(ii) Rearranging Basquin's law (which can be found in the Data Book):

$$\Delta\sigma N_f^{\alpha} = C_1 \ \Rightarrow\ \log\Delta\sigma + \alpha \log N_f = \log C_1$$

Read off any pair of points $(N_{f1}\ ,\ \Delta\sigma_1)$ and $(N_{f2}\ ,\ \Delta\sigma_2)$ from within the high cycle fatigue section of the curve in Fig. 2, and substitute into the equation above. Some suitable values are given in Table 1.

| $N_f$ | Stress range, $\Delta\sigma$ (MPa) |
|---|---|
| $3.16 \times 10^6$ | 308 |
| $1.00 \times 10^6$ | 352 |
| $1.78 \times 10^5$ | 430 |
| $3.16 \times 10^4$ | 525 |

Table 1

Next, eliminating $C_1$ gives: $\alpha = \frac{\log(\Delta\sigma_2) - \log(\Delta\sigma_1)}{\log(N_{f1}) - \log(N_{f2})} = 0.116$

And therefore: $C_1 = \Delta\sigma_1 N_{f1}^{\alpha} = \Delta\sigma_2 N_{f2}^{\alpha} = 1.74 \times 10^3$ MPa. [6]

(iii) Miner's rule of cumulative damage, accounting the three phases of loading, predicts failure to occur when:

$$\frac{N_1}{N_{f1}} + \frac{N_2}{N_{f2}} + \frac{N_3}{N_{f3}} = 1$$

Next, calculate the number of cycles to failure $N_f$ for each phase of loading:
- For phase 1, $\Delta\sigma_1 = 50$ MPa is less than the endurance limit, and so:

$$N_{f1} = \infty, \quad \frac{N_1}{N_{f1}} = 0$$

- Phases 2 ($\Delta\sigma_2 = 300$ MPa) and 3 ($\Delta\sigma_3 = 400$ MPa) lie within the high cycle fatigue regime. The values of $N_f$ can either be (i) read off from the figure in the question (see Fig. 3, below) or (ii) calculated using Basquin's law, with the constants obtained above:

$$\log(N_{f2}) = \frac{1}{\alpha}\log\left(\frac{C_1}{\Delta\sigma_2}\right) = \frac{1}{0.116}\log\left(\frac{1.74 \times 10^3}{300}\right) \quad \therefore N_{f2} = 4.02 \times 10^6$$

$$\log(N_{f3}) = \frac{1}{\alpha}\log\left(\frac{C_1}{\Delta\sigma_3}\right) = \frac{1}{0.116}\log\left(\frac{1.74 \times 10^3}{400}\right) \quad \therefore N_{f3} = 3.34 \times 10^5$$

If the run time is $t_r$, the number of cycles in each phase is $\Omega f_r t_r$, where $f_r$ is the proportion of the run time. Substituting into Miner's rule:

$$\frac{\Omega_2 f_{r2} t_r}{N_{f2}} + \frac{\Omega_3 f_{r3} t_r}{N_{f3}} = 1$$

The lifetime of the component is therefore:

$$t_r = \left(\frac{\Omega_2 f_{r2}}{N_{f2}} + \frac{\Omega_3 f_{r3}}{N_{f3}}\right)^{-1} = \left(\frac{12 \times 0.017}{4.02 \times 10^6} + \frac{6 \times 0.003}{3.34 \times 10^5}\right)^{-1} = 9.55 \times 10^6 \text{ s} = 111 \text{ days}$$

Accuracy of the prediction: Miner's rule is applicable primarily for high cycle fatigue, and is therefore appropriate in this case. However, it ignores the sequence in which the phases of loading occur, applying data for $N_f$ from undamaged specimens to all phases of loading. [10]

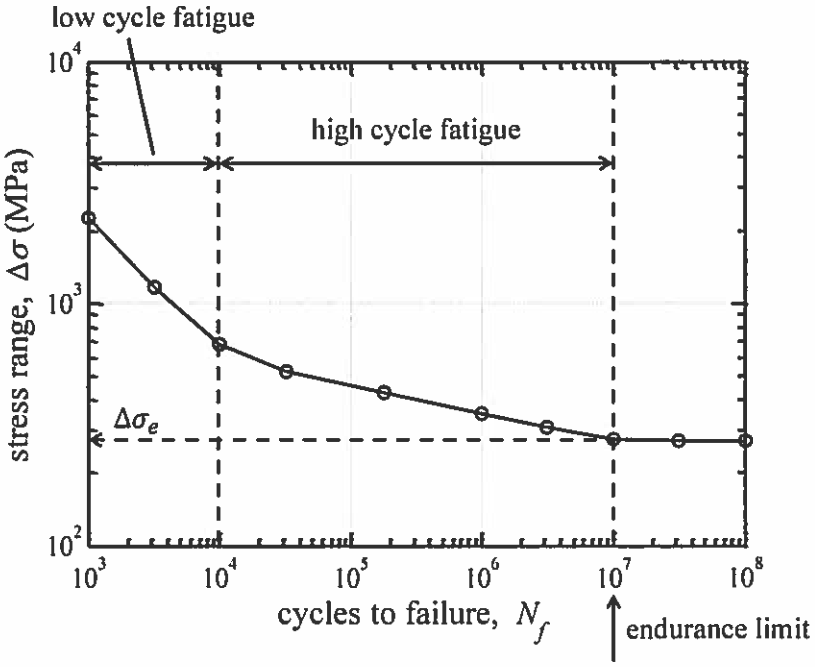

low cycle fatigue
high cycle fatigue
stress range, Δσ (MPa)
Δσe
cycles to failure, Nf
endurance limit
10^4
10^3
10^2
10^3
10^4
10^5
10^6
10^7
10^8

Fig. 2

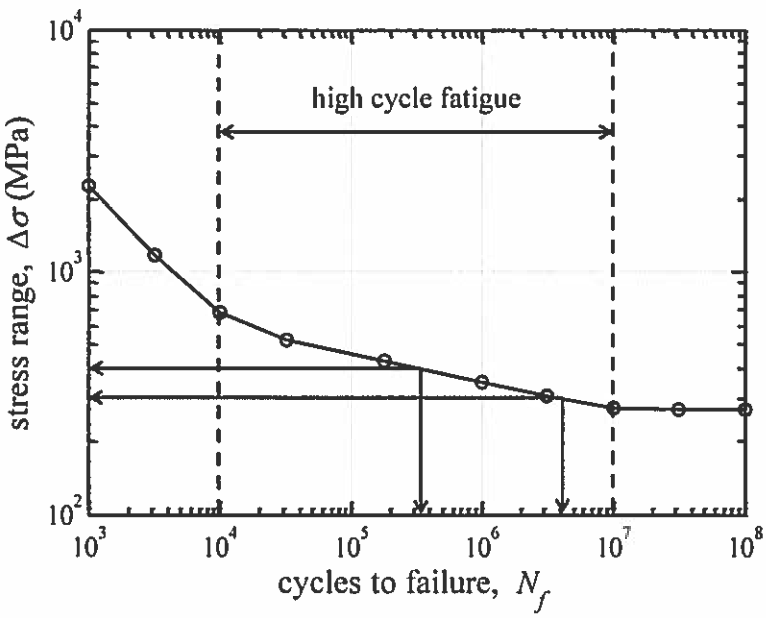

high cycle fatigue
stress range, Δσ (MPa)
cycles to failure, Nf
10^4
10^3
10^2
10^3
10^4
10^5
10^6
10^7
10^8

Fig. 3

12 (**long**)

(a) Stiffness-limited design in bending:

(i) Maximising $\phi$ for minimum weight:
- Increased second moment of area results in greater bending stiffness, for a given cross-sectional area and material (i.e. for a given mass per unit length).
- Greater bending stiffness means reduced deflections for a given weight, or reduced weight for a given deflection. [2]

(ii) Factors that might limit that maximum value of $\phi$ achievable in practice:
- Local buckling failure: A larger $\phi$ generally corresponds to a reduced cross-sectional thickness (e.g. thin-walled tube vs solid circular cross-section). Thinner walls may lead to local buckling under load.
- Manufacturability constraints: For some material choices, shaping into complex cross-sectional shapes may be difficult or impossible. Shaping may also add to the manufacturing costs. [2]

(b) Beam with a solid circular cross-section:

(i) Distributed load due to self weight: $\omega = \rho\pi R^2 g$ N m$^{-1}$.
The second moment of area: $I = \frac{\pi R^4}{4}$
Substitute these results into the Data Book formula for the mid-span deflection of a simply supported beam with a uniformly distributed load:

$$\delta = \frac{5\omega L^4}{384EI} = \frac{5\left(\rho\pi R^2 g\right)L^4}{384E\left(\frac{\pi R^4}{4}\right)} = \frac{5\rho g L^4}{96ER^2} \qquad [4]$$

(ii) Objective: deflection, $\delta$ (above)
Constraint: cost, $C = C_m \times \text{mass} = C_m \pi R^2 L\rho$
Free variable: radius, $R$
*Note that the objective and constraint in this problem are the opposite way round compared to those typically seen (i.e. often mass or cost is the objective, and deflection the constraint). But the analysis methodology is identical in either case.*
Eliminate $R$ from the objective, using the constraint:

$$R^2 = \frac{C}{C_m \pi L\rho} \qquad \Rightarrow \qquad \delta = \frac{5\rho g L^4}{96E\left(\frac{C}{C_m\pi L\rho}\right)} = \frac{5\pi g L^5}{96C}\left(\frac{C_m\rho^2}{E}\right)$$

Therefore, to minimise $\delta$, maximise: $M = \frac{E}{C_m\rho^2}$ [5]

(iii) Substitute the given material properties into the performance index $M$. The values and rankings are given in Table 2. [4]

| Material | $E$ (GPa) | $\rho$ (kg m$^{-3}$) | $C_m$ (£ kg$^{-1}$) | $M = \frac{E}{C_m \rho^2}$ | Rank |
|---|---|---|---|---|---|
| CFRP | 120 | 1500 | 60 | 889 | 3 |
| Al alloy | 70 | 2700 | 2 | 4801 | 2 |
| Nylon | 3 | 1100 | 4 | 620 | 4 |
| Wood | 12 | 600 | 2 | 16667 | 1 |

Table 2

For a cost $C = £500$ and a length $L = 5$ m, calculate the beam radius $R$ and hence the deflection $\delta$, using the equations above. The results are given in Table 3. The material choices that deflect more than 10 mm are: CFRP and nylon. [4]

| Material | $R = \sqrt{\frac{C}{C_m \pi L \rho}}$ (mm) | $\delta = \frac{5\rho g L^4}{96ER^2}$ (mm) |
|---|---|---|
| CFRP | 18.8 | 11.3 |
| Al alloy | 76.8 | 2.09 |
| Nylon | 85.1 | 16.2 |
| Wood | 162.9 | 0.602 |

Table 3

(c) Beam with a thin walled tubular cross-section:

(i) The second moment of area $I$ of the tube can be written in terms of the shape factor using the relationship given in the question: $I = \phi\, I_{\text{ref}}$. If the reference cross-section is a solid circle:

$$I_{\text{ref}} = \frac{\pi R^4}{4} \quad \text{and} \quad A = \pi R^2 \quad \therefore\ I_{\text{ref}} = \frac{A^2}{4\pi} \quad \text{and} \quad I = \frac{\phi A^2}{4\pi} \quad \text{for the tube.}$$ [3]

(ii) Objective: deflection $\delta$, which should now be expressed in terms of the shape factor $\phi$ and the area $A$:

$$\delta = \frac{5\omega L^4}{384EI} = \frac{5(\rho A g)L^4}{384E\left(\frac{\phi A^2}{4\pi}\right)} = \frac{5\pi\rho g L^4}{96EA\phi}$$

Constraint: cost, $C = C_m \times \text{mass} = C_m A L \rho$

Eliminate the free variable $A$ from the objective, using the constraint:

$$\delta = \frac{5\pi\rho g L^4}{96E\left(\frac{C}{C_m L\rho}\right)\phi} = \frac{5\pi g L^5}{96C}\left(\frac{C_m \rho^2}{E\phi}\right) \qquad \therefore\ M = \frac{E\phi}{C_m \rho^2}$$ [6]