Engineering Tripos, Part IA, 2015

Paper 4 Mathematical Methods

Solutions

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Paper 4, I A - Tripos

1) $x_{n+2} - 5x_{n+1} + 4x_n = 0$

$x_0 = 1 ; \quad x_1 = 5$

$x_n = \lambda^n \Rightarrow \lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 5)(\lambda - 1) = 0 \Rightarrow \quad \lambda = 5, 1$

$x^n = A4^n + B1^n$

$$\left.\begin{aligned} x_0 = 1 \quad & 1 = A + B \\ x_1 = 5 \quad & 5 = 4A + B \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 4/3 \\ B &= -1/3 \end{aligned}$$

$$\therefore x^n = \frac{4}{3} 4^n - \frac{1}{3} = \frac{4^{n+1}}{3} - \frac{1}{3}$$

2) a) $$\lim_{x \to 0} \left[\frac{(\sin x - \sin^2 x)}{x^2 - x + \sin^2 x} \right]$$

$$= \lim_{x \to 0} \left[\frac{\sin x + (\cos 2x - 1)/2}{x^2 - x + (1 - \cos 2x)/2} \right]$$

$$\Rightarrow \lim_{x \to 0} \frac{\cos x - \sin 2x}{2x - 1 + \sin 2x} = \underline{\underline{-1}}$$

(b) $\lim_{x \to 0} \left[\frac{\sin x - x}{x^3} - \frac{1}{\cosh x} \right]$

Series:

$$\sin x = x - \frac{x^3}{3!} + O(x^5)$$

$$\sin x - x = -\frac{x^3}{3!} + O(x^5)$$

$$\frac{\sin x - x}{x^3} = -\frac{1}{3!} + O(x^2)$$

$$\therefore \lim_{x \to 0} \left[\frac{\sin x - x}{x^3} - \frac{1}{\cosh x} \right]$$

$$= -\frac{1}{3!} - 1 = \frac{-1}{6} - 1 = \underline{\underline{-7/6}}$$

3) $A = \begin{bmatrix} 8 & 3 \\ 1 & 6 \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} 8-\lambda & 3 \\ 1 & 6-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(6-\lambda) - 3 = 0$$

$$\lambda^2 - 14\lambda + 45 = 0 \Rightarrow \begin{matrix} \lambda_1 = 9 \\ \lambda_2 = 5 \end{matrix}$$

using $\lambda_1 = 9$

$$\begin{bmatrix} 8-\lambda & 3 \\ 1 & 6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-x_1 + 3x_2 = 0$$

$$x_1 - 3x_2 = 0$$

$$\Rightarrow x_1 = 3x_2 \Rightarrow X = \begin{bmatrix} 3 \\ 1 \end{bmatrix} //$$

using $\lambda_2 = 5$

$$3x_1 + 3x_2 = 0 \quad \Rightarrow \quad x_1 = -x_2$$

$$x_1 + x_2 = 0$$

$$\Rightarrow X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} //$$

4) $$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0$$

(a)

$$LC\frac{d^2I}{dt^2} + RC\frac{dI}{dt} + I = 0$$

Auxiliary equation:

$$LC\lambda^2 + RC\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}$$

i) $R^2C^2 - 4LC > 0 \Rightarrow R^2 - \frac{4L}{C} > 0$

$\Rightarrow$ two real roots λ_1, λ_2

$$\Rightarrow I(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

(ii) $R^2 - \frac{4L}{C} = 0$

$\Rightarrow$ two identical roots $\left(-\frac{R}{2L}\right)$

$$\Rightarrow I(t) = (At + B)e^{-\frac{R}{2L}t}$$

(iii) $R^2 - \frac{4L}{C} < 0 \Rightarrow$ roots are complex conjugate ie $\lambda = \alpha \pm i\beta$

$$\therefore I(t) = e^{\alpha t}(A\cos\beta t \pm iB\sin\beta t)$$

(b) Unstable Solutions will result

if λ_1 & λ_2 $\quad R^2 - \frac{4L}{C} > 0$

are possitive ~~real~~ real

also if $\alpha > 0$ in $\lambda = \alpha \pm i\beta$ for $R^2 - \frac{4L}{C} < 0$

exponentially growing ~~soluti~~ oscillatory Solution can result.

(c) $R = 100\,\Omega$, $L = 1 \times 10^{-2}\,H$, $C = 2 \times 10^{-6}$

$$\therefore \lambda = \frac{-2 \times 10^{-4} \pm \sqrt{4 \times 10^{-8} - 8 \times 10^{-8}}}{4 \times 10^{-8}}$$

$$= -0.5 \times 10^4 \pm \frac{i\, 2 \times 10^{-4}}{4 \times 10^{-8}} = -0.5 \times 10^4 \pm i\, 0.5 \times 10^4$$

$$= 0.5 \times 10^4 (-1 \pm i) \Rightarrow \begin{aligned} \alpha &= -0.5 \times 10^4 \\ \beta &= 0.5 \times 10^4 \end{aligned}$$

$$\Rightarrow I(t) = A e^{-0.5 \times 10^4 t} \cos\beta t + B e^{-0.5 \times 10^4 t} \sin\beta t$$

$$\beta = 0.5 \times 10^4 = \omega = 2\pi f$$

$$\therefore \boxed{f = \frac{5}{2\pi} \times 10^3\ Hz.}$$

(d) $I(0) = 1$ $\quad \left.\frac{dI}{dt}\right|_{t=0} = 0$.

$$I = e^{\alpha t}(A\cos\omega t + B\sin\omega t)$$

$$\frac{dI}{dt} = A\alpha e^{\alpha t}\cos\omega t + B\alpha e^{\alpha t}\sin\omega t - A\omega e^{\alpha t}\sin\omega t + B\omega e^{\alpha t}\cos\omega t$$

$\Rightarrow$ @ $t=0$

$$I = 1 = A. \quad \Rightarrow \quad \boxed{A = 1}$$

$$\frac{dI}{dt} = 0 = A\alpha + B\omega \Rightarrow B = \frac{-A\alpha}{\omega} = -\alpha/\omega$$

$$\boxed{B = -\alpha/\omega}$$

$$\alpha = -0.5\times10^4; \quad \omega \equiv \beta = 0.5\times10^4$$

$\Rightarrow$ $B = 1.$

$\therefore$ Complete Solution is

$$\boxed{I(t) = e^{-5\times10^3 t}\left[\cos(5\times10^3 t) + \sin(5\times10^3 t)\right]}$$

5)

$$\frac{|z-i|}{|z+i|} = a \qquad a = \text{Const}$$

$$\left[\frac{|z-i|}{|z+i|}\right]^2 = a^2$$

$$\frac{|z-i|^2}{|z+i|^2} = a^2$$

$$\frac{|x+i(y-1)|^2}{|x+i(y+1)|^2} = a^2$$

$$x^2 + (y-1)^2 = a^2\left(x^2 + (y+1)^2\right)$$

$$x^2(1-a^2) + y^2(1-a^2) - 2y(1+a^2) = a^2 - 1$$

$$\therefore \quad x^2 + y^2 - 2y\frac{(1+a^2)}{(1-a^2)} = -1$$

$$\cancel{x^2 + \left(y - 2y\frac{(1+a^2)}{(1-a^2)} + \frac{(1+a^2)^2}{(1-a^2)^2}\right) = \frac{(1+a^2)^2}{(1-a^2)^2} - 1}$$

i) $$x^2 + \left(y - \frac{(1+a^2)}{(1-a^2)}\right)^2 = \underbrace{\frac{(1+a^2)^2}{(1-a^2)^2} - 1}_{\text{Constant}}$$

~~Is the locus.~~

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$a < 1$ for real values of x an y.
i.e sq.rt of R.H.S. constant must be real.

$$\therefore \frac{(1+a^2)^2}{(1-a^2)^2} > 1$$

$$1+a^2 > 1-a^2$$

$$\therefore a > 0$$

Also note that

$$x = \pm\sqrt{\frac{2(1+a^2)}{(1-a^2)}y - y^2 - 1}$$

$\therefore y > 0 \quad a > 1$ for real x

(iii) $a = 0.9 \qquad c = \frac{(1+a^2)}{1-a^2} = \frac{1.81}{0.19} = 9.526$

$\therefore y^2 - 2 \times 9.526\, y + 1 = 0 \qquad x = 0$ at y intersect.

$$\therefore y = \frac{19.05 \pm \sqrt{(19.05)^2 - 4}}{2}$$

$$= 9.526 \pm 9.47$$

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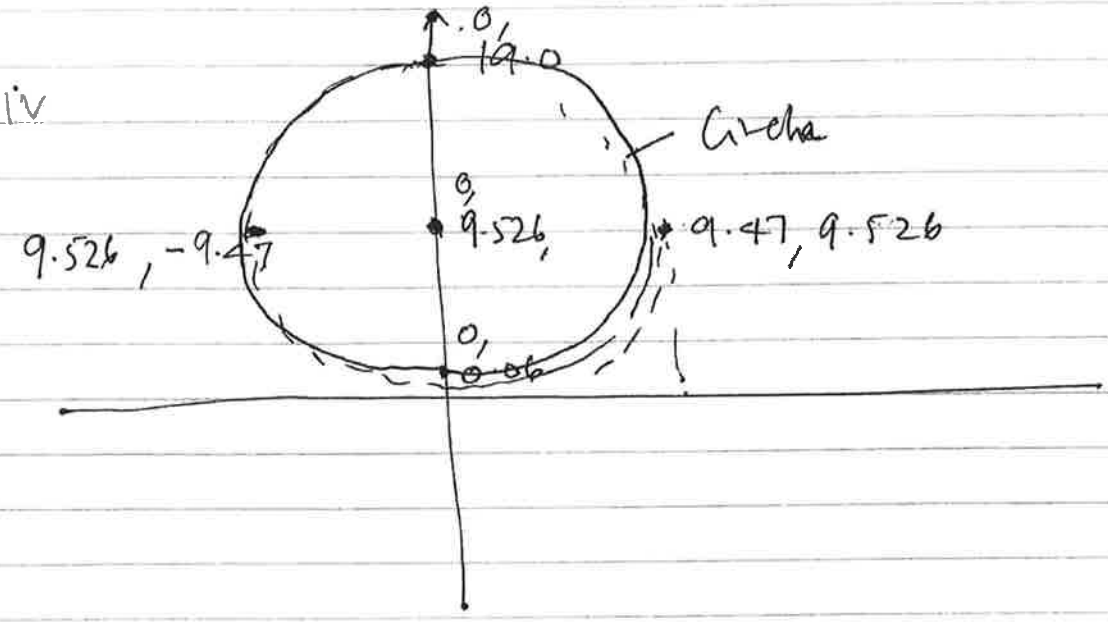

When $y = c$ $\qquad x^2 = c^2 - 1$

$$x = \pm\sqrt{c^2-1} = \pm 9.47$$

Also note that $\dfrac{19.0 - 0.06}{2} = 9.47$.

Therefore locus are circle with centre at $c = 9.526$ and $r = 9.47$

$$x^2 + (y - 9.526)^2 = 9.47^2$$

$$x^2 + (y - c)^2 = r^2$$

6) (Short)

Section B

$$\frac{dx^2}{dt} + 2\frac{dx}{dt} + x = 1$$

$x(0) = 0$

$\frac{dx}{dt}(0) = 0 = \dot{x}(0)$

Taking Laplace transform on both sides (from data book)

$$s^2 X - s\cancel{x(0)}^{0} - \cancel{\dot{x}(0)}^{0} + 2sX - \cancel{x(0)}^{0} + X = \frac{1}{s}$$

$$\Rightarrow (s^2 + 2s + 1) X(s) = \frac{1}{s}$$

$$(s+1)^2 X(s) = \frac{1}{s} \Rightarrow X(s) = \frac{1}{s(s+1)^2}$$

$$\frac{1}{s(s+1)^2} = \frac{1}{s} + \frac{As+B}{(s+1)^2}$$

$$\Rightarrow 1 = (A+1)s^2 + (B+2)s + 1$$

$$\Rightarrow A = -1, \quad B = -2$$

$$\therefore X(s) = \frac{1}{s} - \frac{s+2}{(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}$$

Taking inverse transform from the data book

$$x(t) = 1 - e^{-t} - te^{-t} = 1 - e^{-t}(1+t)$$

$$\therefore \boxed{x(t) = 1 - (1+t)e^{-t}}$$

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7 (long)

(a) $$y = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Change of variable $t-\tau = \psi$

$\Rightarrow \tau = t-\psi$

$d\tau = -d\psi$

$$\Rightarrow y(t) = -\int_t^0 f(t-\psi)\, g(\psi)\, d\psi$$

$$= \int_0^t g(\psi)\, f(t-\psi)\, d\psi$$

Since ψ is a dummy variable

$$= \int_0^t g(\tau)\, f(t-\tau)\, d\tau.$$

Thus, $$\int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t g(\tau)\, f(t-\tau)\, d\tau.$$

7) (long)

b) (i) $\alpha \frac{dy}{dt} + y = f(t)$

$\Rightarrow$ C.F : $y(t) = A e^{-t/\alpha}$

P.I. $y(t) = 1$

$\Rightarrow \quad y(t) = 1 + A e^{-t/\alpha}$

y has to be continuous @ $t=0 \Rightarrow y(0^+) = y(0^-)$

& $y(t) = 0$ for $t < 0$

$\Rightarrow \quad 1 + A = 0 \quad \Rightarrow \quad \boxed{A = -1}$

$\therefore \boxed{y(t) = 1 - e^{-t/\alpha}}$ Step response.

(ii)

Impulse response is $\frac{d}{dt}$ (step response)

$$\Rightarrow \quad g(t) = \begin{cases} \frac{1}{\alpha} e^{-t/\alpha} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(iii) $$f(t) = \begin{cases} \frac{1}{\beta} e^{-t/\beta} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\beta < \alpha.$

$$y(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \frac{1}{\beta} e^{-\tau/\beta} \frac{1}{\alpha} e^{-\frac{(t-\tau)}{\alpha}} d\tau$$

$$= \frac{1}{\alpha\beta} e^{-t/\alpha} \int_0^t e^{\tau \frac{(\beta-\alpha)}{\alpha\beta}} d\tau$$

$$\Rightarrow \boxed{y(t) = \frac{e^{-t/\alpha} - e^{-t/\beta}}{\alpha - \beta}}$$

The response is

$$y(t) = \begin{cases} \frac{e^{-t/\alpha} - e^{-t/\beta}}{\alpha-\beta} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

As $\beta \to 0$ $y(t) \to g(t)$ impulse response

$\Rightarrow$ $f(t)$ behaves like an impulse function.

8) (Short)

1^{st} person can have any birthday

2^{nd} person will have a day with probability $\frac{(d-1)}{d}$

3^{rd} person ... probability $\frac{(d-2)}{d}$

4^{th} " " $\left(\frac{d-3}{d}\right)$

$\therefore$ for n^{th} person $\frac{(d-n+1)}{d}$

So, the probability for all n people to have different birthday is

$$P_n = \frac{(d-1)}{d} \frac{(d-2)}{d} \cdots \frac{(d-n+1)}{d}$$

$$= \frac{(d-1)!}{(d-n)!\, d^{n-1}} = \frac{d!}{(d-n)!\, d^n}$$

$$\Rightarrow \boxed{P_n = \frac{d!}{(d-n)!\, d^n}}$$ as required.

9) $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

$$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150 = 6(x^2+y^2-25)$$

$$\frac{\partial^2 f}{\partial x^2} = 12x$$

$$\frac{\partial f}{\partial y} = 12xy - 9y^2 = 3y(4x-3y)$$

$$\frac{\partial^2 f}{\partial y^2} = 12x - 18y \; ; \qquad \frac{\partial^2 f}{\partial x \partial y} = 12y$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x^2+y^2 = 5^2 \quad \text{—} \quad (1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0 \text{ or } y = \frac{4}{3}x. \quad -(2)$$

$y=0$ in (1) $\Rightarrow$ $x = \pm 5$

$y = \frac{4}{3}x$ in (1) $\Rightarrow$ $x = \pm 3 \Rightarrow y = \pm 4$

$\therefore$ The ~~pair~~ Sationary points are

$(5,0)$, $(-5,0)$, $(3, 4)$ and $(-3 \; -4)$

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At these four points

	$(5,0)$	$(-5,0)$	$(3,4)$	$(-3,-4)$
$\frac{\partial^2 f}{\partial x^2}$	60	-60	36	-36
$\frac{\partial^2 f}{\partial y^2}$	60	-60	-36	36
$\frac{\partial^2 f}{\partial x \partial y}$	0	0	48	-48
$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	+ve	+ve	−ve	−ve

(from Data book)

$\Rightarrow$ $(5,0)$ is minium

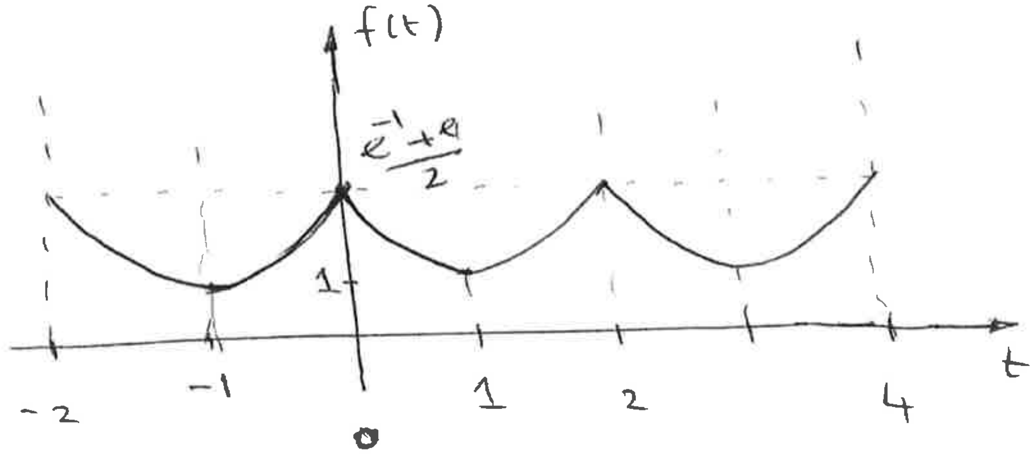
$(-5,0)$ is maximum

$(3,4)$ & $(-3,-4)$ are Saddle points.

10) (Long)

$$f(t) = \cosh(t-1) \qquad 0 \le t \le 1$$

(a)



(b) even function with T, Period = 2 (−1 to 1)

from data book

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n t}{T}$$

with $a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos\left(\frac{2\pi n t}{T}\right) dt$

$$\Rightarrow \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\pi n t)$$

$$a_n = 2\int_0^1 f(t)\cos(n\pi t)\, dt$$

$$a_0 = 2\int_0^1 f(t)\, dt = +2\sinh(t-1)\Big|_0^1$$

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$$a_0 = -2\sinh(-1) = 2\sinh(1)$$

$$a_n = 2\int_0^1 \cosh(t-1)\cos(n\pi t)\,dt$$

$$= 2\int_0^1 \cos(n\pi t)\, d(\sinh(t-1))$$

$$= 2\sinh(t-1)\cos n\pi t\Big|_0^1 + 2n\pi\int_0^1 \sinh(t-1)\sin n\pi t\,dt$$

$$= 2\sinh(1) + 2n\pi\int_0^1 \sin n\pi t\, d(\cosh(t-1))$$

$$= 2\sinh(1) + 2n\pi\left\{\cosh(t-1)\sin n\pi t\Big|_0^1 - n\pi\int_0^1 \cosh(t-1)\cos n\pi t\,dt\right\}$$

$$= 2\sinh(1) - 2n^2\pi^2\int_0^1 \cosh(t-1)\cos n\pi t\,dt$$

$$a_n = 2\sinh(1) - n^2\pi^2 a_n$$

$$\Rightarrow \quad a_n = \frac{2\sinh(1)}{1+n^2\pi^2}$$

$$\therefore f(t) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{\cos n\pi t}{(1+n^2\pi^2)}$$

(c) for $t=0$

$$\cosh(1) = \sinh(1) + 2\sinh(1) \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}$$

$$\frac{e}{2} + \frac{1}{2e} = \frac{e}{2} - \frac{1}{2e} + \left(e - \frac{1}{e}\right) \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}$$

$$\frac{1}{e} = \frac{e^2-1}{e} \sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{1+n^2\pi^2} = \frac{1}{e^2-1}}$$ as required.

2015 Paper 4: Mathematical Methods

Solutions to Section C

11. **Data structures**

(a) Each vertex requires 24 bytes and each triangle requires 12 bytes. There are 5580 vertices and 11119 triangles. The entire mesh therefore requires $24 \times 5580 + 12 \times 11119 = 267348$ bytes. [3]

(b) A suitable code segment would look something like this.

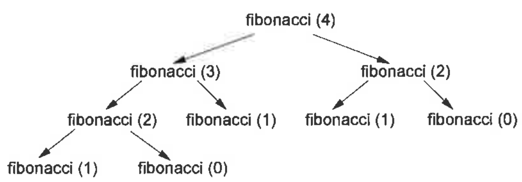
```
vertex a, b, c; // vertices of the first triangle
vertex centroid;

a = v_list[t_list[0].v1];
b = v_list[t_list[0].v2];
c = v_list[t_list[0].v3];

centroid.x = (a.x + b.x + c.x)/3.0;
centroid.y = (a.y + b.y + c.y)/3.0;
centroid.z = (a.z + b.z + c.z)/3.0;
```

[7]

Assessor's remarks: In (a), the vast majority of candidates were able to calculate the memory requirements of the triangular mesh. The purpose of (b) was to test manipulation of data structures, i.e. how to access the first triangle's vertex coordinates. Unfortunately, there was a prerequisite requirement to know how to calculate a triangle's centroid, and this turned out to be a significant hurdle for many candidates. Only around half realised it was simply the average of the three vertex coordinates. Others made no attempt at (b), or recalled geometrical constructions involving the intersection of bisectors and then attempted to embody these constructions in messy C++. Of those candidates who did propose the simple vertex average, around half produced correct C++ implementations.

12. **Algorithmic complexity and numerical accuracy**

(a) For the recursive Method 1, and taking $n = 4$ as an example, the pattern of function calls is as follows:



There are $O(n)$ levels of recursion, and the number of function calls doubles at each level (approximately — the call tree is not symmetric, so "doubles" is the worst case). The complexity is therefore $O(2^n)$ (upper bound) and the algorithm is practically useless.

Method 2 is the closed form solution with execution time that does not depend on n (assuming a constant-time implementation of `pow()`). Its complexity is therefore $O(1)$. [5]

(b) Method 1 uses integer arithmetic and will be perfectly accurate until f_i exceeds the range of a signed integer. Method 2 uses floating point arithmetic and we can therefore expect truncation errors. For example, the 46th Fibonacci number is 1836311903, but Method 2 returns 1836313576.03. Method 2 could be improved, but not made perfect, by switching to double precision variables. [5]

Assessor's remarks: In (a), around half the candidates made wild, unsubstantiated guesses at the algorithmic complexities, scoring zero marks. For the others, the exponential complexity of the recursive function was more problematic than the $\mathcal{O}(1)$ complexity of the closed form solution. It was pleasing to see many of the more thoughtful candidates querying the time complexity of the C++ `pow()` function: candidates who answered $\mathcal{O}(n)$ for the closed form solution, on the grounds that `pow()` performs repeated multiplications, were awarded full marks even though the standard x86 implementation of `pow()` runs in constant time. Other candidates even realised that $\mathcal{O}(2^n)$ is not a tight bound for the complexity of the recursive function, and went on to derive the correct tight bound. In (b), most candidates knew that the integer calculations would be perfect whereas the floating point calculations would not. Unfortunately, others stated that integers, since they ignored the fractional parts of the numbers, could not possibly be more accurate than floats. And others suggested that "mantissa shifting" would introduce errors into integer summations.

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