IA Tripos Paper 4 – Mathematical Methods

2016 – CRIBS

Section A:

(1) (a) Use l'Hopital's rule: twice on $f(x) = (x\cos x - \sin x)/x^3$

First $= (\cos x - \sin x - \cos x)/3x2 = -\sin x/3x2$

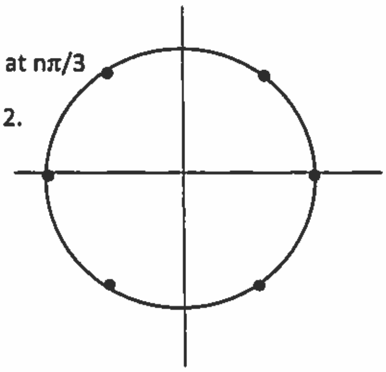
Second $= [-\cos x/3 = -1/3$

(b) Use l'Hopital's rule once on $g(x) = \sin(\pi x)/(x^2-16)$

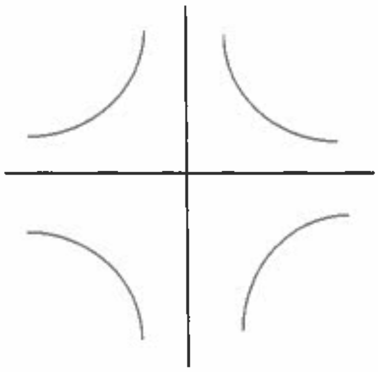
$= \pi \cos(\pi x) / 2x = \pi\cos(-4\pi)/ -8 = -\pi/8$

2: (a) Solutions at $n\pi/3$

Circle radius 2.



(b) $\text{Im}(z^2) = 2xy$



(3) (a) In any pairs, the vectors **a**, **b**, **c**, satisfy **u.v** = 0 so they are orthogonal.

Their magnitudes are $3\sqrt{2}$, 3, $3\sqrt{2}$ respectively so the volume is 56.

(b) by inspection the order they are in forms a right handed set as does any cyclic permutation.

(4) (a) By inspection **U** is a rotation of θ about the origin. For $\theta = \pi/4$ we get a 45° rotation.

$[1,1] \rightarrow [\sqrt{2}, 0]$, $[1,2] \rightarrow [3/\sqrt{2}, 1/\sqrt{2}]$, $[2,2] \rightarrow [2\sqrt{2}, 0]$, $[2,1] \rightarrow [3/\sqrt{2}, -1/\sqrt{2}]$

Plot points and observe they form a square of unit side.

(b) $\mathbf{U}^4$ = 4 rotations of -45° =one rotation of -1, so $\mathbf{U}^4$=-$\mathbf{I}$, can do it by numbers and multiply out the matrices. By inspection, rotation of $-\theta$ gives a transpose of **U** so $\mathbf{U}^{-1} = \mathbf{U}^T$

(c) $||\mathbf{A}-\lambda\mathbf{I}||$=0 implies $\lambda^3 - 3\lambda a^2 - 2a^3$=0 or λ= -a, -a, 2a.

largest eigenvalue: λ = 2a Eigenvector: $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

Other two eigenvectors in the plane perpendicular to {1,1,1}.

(5) (a) $dN/dt = -\lambda N$ integrates up to $N(t)=N_0\exp(-\lambda t)$

After time $T_{1/2}$ we have half the number of atoms left so $N(T_{1/2})/N_0 = \exp(-\lambda\, T_{1/2}) = 0.5$ and

So : $T_{1/2} = \ln(2)/\lambda$.

(b) If the N_1 species is stable, i.e. has a very long lifetime, then the N_1's being produced are all those that decayed, $N_1(t) + N(t) = N_0$, so $N_1(t) = N_0\,(1 - \exp(-\lambda t))$. This satisfies the boundary conditions that there are no N_1 species at t=0 and they are all N_1 species at very large time.

(c) If the N_1 species are unstable, then the rate of growth of N_1 is determined by the rate of formation of N_1 from N_0 minus the loss of N_1 to the daughter product N_2

So: $dN_1(t)/dt = \lambda N_0\exp(-\lambda t) - \lambda^* N_1(t)$ =

Integrate $dN_1(t)/dt + \lambda^*\, N_1(t) = \lambda N_0\exp(-\lambda t)$

Particular integral: $N_1(t) = A\exp(-\lambda t)$: $-\lambda A + \lambda^* A = \lambda N_0$, and so $A= -\lambda N_0/(\lambda-\lambda^*)$

No: $N1(t) = A\exp(-\lambda t) + B\exp(-\lambda^* t)$, and given that $N1(t=0) = 0$, $B=-A$

$N_1(t) = [N_0/(\lambda^*-\lambda)][\exp(-\lambda t) - \exp(-\lambda^* t)]$.

(d) $dN_1(t)/dt= 0$ when $\lambda\exp(-\lambda t) = \lambda^*\exp(-\lambda^* t)$ or $t^* = \ln(\lambda/\lambda^*)/(\lambda-\lambda^*)$

Note that both numerator and denominator are positive if $\lambda>\lambda^*$, and both are negative if $\lambda<\lambda^*$, so t* is always positive.

Section - B

Q6 (Short)

$$\frac{d^2x}{dt^2} - 4x = \sinh t \qquad x(0) = 0, \quad \left.\frac{dx}{dt}\right|_{x=0} = 0.$$

$$s^2 X - 4X = \frac{1}{s^2-1}$$

$$X(s) = \frac{1}{(s^2-1)(s^2-4)} = \frac{As+B}{(s^2-1)} + \frac{Cs+D}{(s^2-4)}$$

$$\Rightarrow As^3 + Bs^2 - 4As - 4B + Cs^3 + Ds^2 - Cs - D = 1$$

$$\Rightarrow A + C = 0 \qquad -4A - C = 0. \Rightarrow \boxed{A = C = 0}$$

$$B + D = 0 \qquad -4B - D = 1$$

$$\Rightarrow \boxed{B = -1/3 \;\; \& \;\; D = 1/3}$$

$$\therefore X(s) = \frac{-1/3}{(s^2-1)} + \frac{1/3}{(s^2-4)}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{6}\sinh 2t - \frac{1}{3}\sinh t.}$$

Q7 (long)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = f(t) \; ; \quad y(0) = \dot{y}(0) = 0 .$$

(a) Step response: $\quad f(t) = 1 \qquad t \geq 0 .$

C.F.: $\quad y = e^{\lambda t}, \qquad f(t) = 0 .$

$$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= -1 \pm 2i$$

$$\therefore \; y(t) = e^{-t}\left[A\cos 2t + B\sin 2t\right]$$

P.I.: $\quad y = \alpha \Rightarrow 5\alpha = 1 \Rightarrow \alpha = 1/5$

$\therefore$ Step response is

$$y(t) = \frac{1}{5} + e^{-t}\left[A\cos 2t + B\sin 2t\right]$$

$$\dot{y}(t) = e^{-t}\left[(2B-A)\cos 2t + (-B-2A)\sin 2t\right]$$

$$y(0) = 0 \Rightarrow \frac{1}{5} + A = 0 \Rightarrow \boxed{A = -1/5}$$

$$\dot{y}(0) = 0 \Rightarrow 2B - A = 0 \Rightarrow \boxed{B = -1/10}$$

$\therefore$ Step response is

$$y(t) = \frac{1}{5} - e^{-t}\left[\frac{\cos 2t}{5} + \frac{\sin 2t}{10}\right]$$

Impulse response is the derivative of the step response.

$$g(t) = \dot{y}(t) = \frac{\sin 2t}{2} e^{-t}.$$

(b) Now for $f(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t \geq 0. \end{cases}$

$$y(t) = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t e^{-(t-\tau)} \sin 2(t-\tau)\, e^{-\alpha\tau}\, d\tau$$

$$= \frac{e^{-t}}{2}\int_0^t e^{(1-\alpha)\tau} \sin 2(t-\tau)\, d\tau$$

integrate by parts:

$$= \frac{e^{-t}}{2}\int_0^t \sin 2(t-\tau)\, d\left[\frac{e^{(1-\alpha)\tau}}{(1-\alpha)}\right]$$

$$= \frac{e^{-t}}{2}\left\{ \frac{e^{(1-\alpha)\tau}}{(1-\alpha)} \sin 2(t-\tau)\Big|_0^t + \frac{2}{(1-\alpha)}\int_0^t e^{(1-\alpha)\tau} \cos 2(t-\tau)\, d\tau \right]$$

(6)

$$= \frac{e^{-t}}{2}\left\{ \frac{-\sin 2t}{(1-\alpha)} + \frac{2}{(1-\alpha)} \int_0^t \cos 2(t-\tau)\, e^{(1-\alpha)\tau}\, d\tau \right\}$$

$$= \frac{e^{-t}}{2}\left\{ \frac{-\sin 2t}{(1-\alpha)} + \frac{2}{(1-\alpha)} \int_0^t \cos 2(t-\tau)\, d\left[\frac{e^{(1-\alpha)\tau}}{(1-\alpha)}\right] \right\}$$

$$= -\frac{e^{-t}\sin 2t}{2(1-\alpha)} + \frac{e^{-t}}{2}\left\{ \frac{2}{(1-\alpha)^2}\left[e^{(1-\alpha)\tau}\cos 2(t-\tau)\Big|_0^t - 2\int_0^t e^{(1-\alpha)\tau}\sin 2(t-\tau)\, d\tau \right] \right\}$$

$$= -\frac{e^{-t}\sin 2t}{2(1-\alpha)} + \frac{e^{-t}}{(1-\alpha)^2}\left[e^{(1-\alpha)t} - \cos 2t \right] - \frac{4}{(1-\alpha)^2}\, \underbrace{\frac{e^{-t}}{2}\int_0^t e^{(1-\alpha)\tau}\sin 2(t-\tau)\, d\tau}_{y(t)}$$

$$\Rightarrow \left(1 + \frac{4}{(1-\alpha)^2}\right) y(t) = \frac{-e^{-t}\sin 2t}{2(1-\alpha)} + \frac{e^{-\alpha t} - e^{-t}\cos 2t}{(1-\alpha)^2}$$

$$\therefore\ y(t) = \left[1 + \frac{4}{(1-\alpha)^2}\right]^{-1}\left[\frac{e^{-\alpha t} - e^{-t}\cos 2t}{(1-\alpha)^2} - \frac{e^{-t}\sin 2t}{2(1-\alpha)} \right]$$

or

$$y(t) = \left[1 + \frac{(1-\alpha)^2}{4}\right]^{-1}\left[\frac{e^{-\alpha t} - e^{-t}\cos 2t}{4} - \frac{(1-\alpha)e^{-t}\sin 2t}{8} \right]$$

If the integration by parts is done using $\sin 2(t-\tau)$

when $\alpha \to 0$

both of the above solutions give

$$y(t) = \frac{1}{5} - e^{-t}\left[\frac{\cos 2t}{5} + \frac{\sin 2t}{10}\right]$$

Q8 (Short)

(a)

$$P_0 = 1 - (P_1 + P_2 + P_3 + \cdots)$$

$$= 1 - (\alpha p^1 + \alpha p^2 + \alpha p^3 + \cdots)$$

$$\boxed{P_0 = 1 - \frac{\alpha p}{1-p}}$$

(Geometric Series – from data book)

as required.

(b)

$$(1-x)^{-(n+1)} = 1 - (n+1)(-x) + \frac{-(n+1)(-n-2)}{2!}(-x)^2 + \cdots$$

(Binomial expansion from the data book)

$$= 1 + (n+1)x + \frac{(n+1)(n+2)}{2!}x^2 + \cdots$$

$$\frac{1}{(1-x)^{n+1}} = \binom{n}{n} + \binom{n+1}{n}x + \binom{n+2}{n}x^2 + \cdots$$

as required.

Q9 (Short)

$$f(x,y) = 5x^3 - 2y^3 - 6x^2y + 54y$$

$$\frac{\partial f}{\partial x} = 15x^2 - 12xy = 3x(5x - 4y) = 0 \quad \Rightarrow \boxed{\begin{array}{l} x = 0 \\ x = \frac{4}{5}y \end{array}}$$

$$\partial^2 f / \partial x^2 = 30x - 12y$$

$$\frac{\partial f}{\partial y} = -6y^2 - 6x^2 + 54 = -6(x^2+y^2) + 54 = 0$$

$$\Rightarrow x^2 + y^2 = \frac{54}{6} = 3^2 \quad — \; (1)$$

$$\frac{\partial^2 f}{\partial y^2} = -12y$$

$x = 0$ in (1) $\Rightarrow y = \pm 3$

$x = \frac{4}{5}y$ in (1) $\Rightarrow y = \pm \frac{15}{\sqrt{41}}$

$$\frac{\partial^2 f}{\partial x \partial y} = -12x$$

Stationary points are

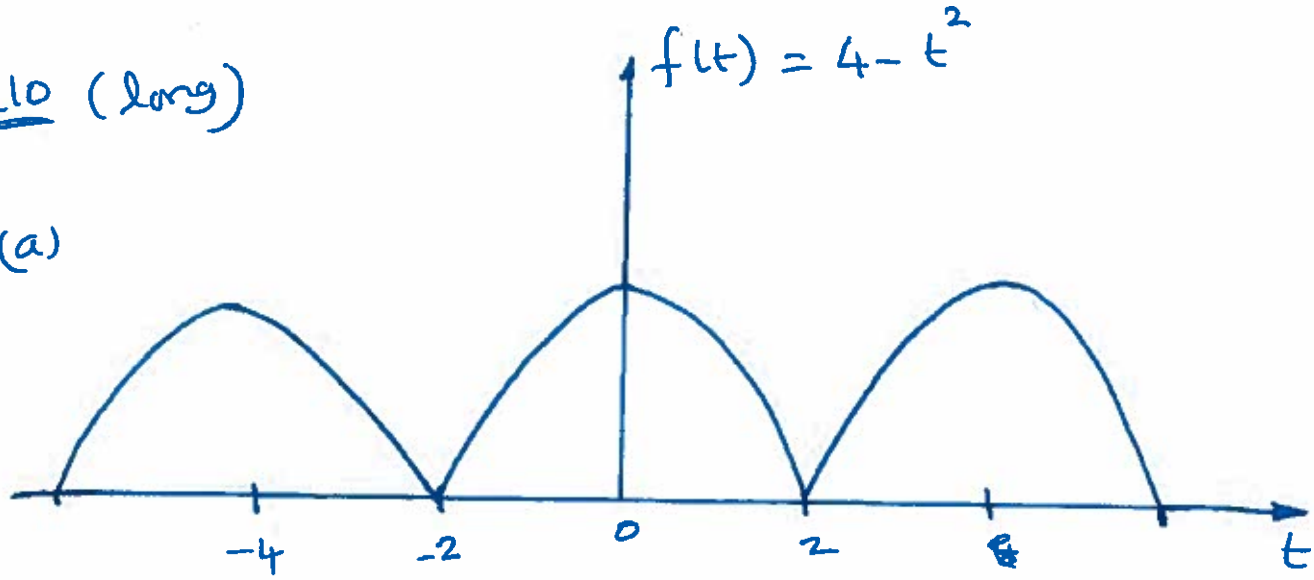
$(0, 3)$ $(0, -3)$, $\left(\frac{12}{\sqrt{41}}, \frac{15}{\sqrt{41}}\right)$, $\left(-\frac{12}{\sqrt{41}}, \frac{-15}{\sqrt{41}}\right)$

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2$$

	$\partial^2 f/\partial x^2$	$\partial^2 f/\partial y^2$	$\frac{\partial^2 f}{\partial x \partial y}$	Δ	
$(0, 3)$	-36	-36	0	$+ve$	MAximum
$(0, -3)$	36	36	0	$+ve$	Minimum.
$\left(\frac{12}{\sqrt{41}}, \frac{15}{\sqrt{41}}\right)$	$\frac{180}{\sqrt{41}}$	$-\frac{180}{\sqrt{41}}$	$\frac{-144}{\sqrt{41}}$	$-ve$	Saddle
$\left(-\frac{12}{\sqrt{4}}\; \frac{-15}{\sqrt{41}}\right)$	$\frac{-180}{\sqrt{41}}$	$\frac{180}{\sqrt{41}}$	$\frac{144}{\sqrt{41}}$	$-ve$.	Saddle.

Q10 (long)

(a)



$T = 4, \quad (-2, \text{to } 2)$

(b)
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} \qquad \text{(even function)}$$

$$a_n = \frac{2}{T} \int_{-2}^{2} f(t) \cos\left(\frac{2\pi n t}{T}\right) dt$$

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(t)\, dt = \int_0^2 f(t)\, dt$$

$$= \left[4t - \frac{t^3}{3}\right]_0^2 = \frac{16}{3}$$

$$\boxed{a_0 = \frac{16}{3}}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} (4-t^2)\cos\left(\frac{n\pi t}{2}\right)\, dt$$

integrate by parts:

$$= \frac{1}{2}\left[\frac{8}{\pi n}\sin\left(\frac{n\pi t}{2}\right)\right]_{-2}^{2} - \frac{1}{2}\int_{-2}^{2} t^2 \cos\left(\frac{n\pi t}{2}\right) dt$$

$$= -\frac{1}{2}\int_{-2}^{2} t^2 \; d\left[\frac{\sin(n\pi t/2)}{n\pi/2}\right]$$

$$= -\frac{1}{n\pi}\left[t^2 \sin\left(\frac{n\pi t}{2}\right)\right]_{-2}^{2} + \frac{1}{n\pi}\int_{-2}^{2} \sin\left(\frac{n\pi t}{2}\right)(2t)\,dt$$

$$= \frac{2}{n\pi}\int_{-2}^{2} t \; d\left[\frac{-\cos(n\pi t/2)}{n\pi/2}\right]$$

$$= -\frac{4}{n^2\pi^2}\left[t\cos\left(\frac{n\pi t}{2}\right)\right]_{-2}^{2} + \frac{4}{n^2\pi^2}\int_{-2}^{2} \cos\frac{n\pi t}{2}\, dt$$

$$= -\frac{4}{n^2\pi^2}\left[2\cos n\pi + 2\cos n\pi\right] + \frac{4}{n^2\pi^2}\,\frac{2}{n\pi}\cancelto{0}{\left[\sin\left(\frac{n\pi t}{2}\right)\right]_{-2}^{2}}$$

$$a_n = -\frac{16}{n^2\pi^2}(-1)^n = \frac{16}{n^2\pi^2}(-1)^{n+1} \qquad n \neq 0.$$

$$\therefore\ f(t) = \frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos\left(\frac{n\pi t}{2}\right)$$

as required.

(c) $$f(0) = 4 = \frac{8}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\boxed{1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.}$$

as required.

Engineering Tripos Part IA FIRST YEAR

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Solutions to Section C

11. **Data structures**

(a) Any one of the following four lines is acceptable:

```
board[0][0].type = Rook; board[0][0].colour = White;
board[0][7].type = Rook; board[0][7].colour = White;
board[7][0].type = Rook; board[7][0].colour = White;
board[7][7].type = Rook; board[7][7].colour = White;
```
[2]

(b)

```
for (int i=0; i<8; i++)
 for (int j=0; j<8; j++)
  if (board[i][j].type == Pawn) cout << i << ' ' << j << endl;
```
[4]

(c)

```
int n = 0;
for (int i=0; i<8; i++)
 for (int j=0; j<8; j++)
  if ((board[i][j].type != None) and (board[i][j].colour == Black)) n = n+1;
cout << n << endl;
```
[4]

12. **Algorithmic complexity and numerical accuracy**

(a) `f1` is clearly $O(n)$. The complexity of `f2` depends on the implementation of exp, though we can expect this to be fast on common architectures. In fact, the standard x86 implementation of `pow` is $O(1)$ and appears to use more or less this method (there are single x86 machine code instructions for exponentiation and scaled logarithms). [4]

(b) The program will display a small, non-zero number. This is because `f1` will calculate 7^2 precisely, whereas `f2` will lose some precision in the floating point representation of $\log(7.0)$. The main program does, in fact, display `7.62939e-06`. [3]

(c) `f1` will fail when n is negative, `f2` will fail when a is zero or negative. Both will fail when a and/or n are sufficiently large to cause a floating point overflow. [3]

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June 2016