Paper 4: Mathematical Methods

Solutions to 2019 Sections B and C

9. **Differential equations**

(a) Characteristic equation: $\lambda^2 + (2+\alpha)\lambda + 2\alpha = (\lambda+2)(\lambda+\alpha) = 0$ with solutions $\lambda = -2$ and $\lambda = -\alpha$. General solution when $\alpha = 3$ therefore $x(t) = Ae^{-2t} + Be^{-3t}$, for $A, B \in \mathbb{R}$. [10]

(b) Two cases, depending on whether $\alpha = 2$. First $\underline{\alpha \neq 2}$:
Solution from part (a) with additional constraint that $x(0) = A + B = 1$, therefore $x(t) = Ae^{-2t} + (1-A)e^{-\alpha t}$, for $A \in \mathbb{R}$.
$\underline{\text{case } \alpha = 2}$: characteristic equation only has one solution $\lambda = 2$. Solution therefore $(At + B)e^{-2t}$. The constraint $x(0) = 1$, yields $B = 1$, so general solution $x(t) = (At+1)e^{-2t}$, for $A \in \mathbb{R}$. [10]

(c) For the inhomogeneous case we need to find a particular integral. The right hand side inspires to try $x(t) = Ct + D$. Plugging into the differential equation we obtain $4C + 4Ct + 4D = 1 + t$, with solutions $C = 1/4$ and $D = 0$, i.e. the particular integral $x(t) = t/4$. The constraint $x(0) = 0$ gives $B = 0$, so the general solution is $x(t) = Ate^{-2t} + t/4$, for $A \in \mathbb{R}$. [10]

Assessors' remarks: This question was a straightforward differential equation question, with a second order differential equation with a free parameter, and questions including homogeneous and inhomogeneous versions, as well as various types of boundary conditions. Students generally did very well. The only persistent issue, was that for part (b), for a particular setting of the free parameter, the characteristic equations no longer has two distinct solutions. A large majority of the students failed to deal correctly with this, even when part (c) explicitly asked for examination of that case (which the majority then proceeded to do correctly).

10. **Difference equations and eigenvectors**

(a) Setting

$$A = \begin{bmatrix} \alpha & \frac{1}{2} \\ 1-\alpha & \frac{1}{2} \end{bmatrix},$$

recovers the desired equations for the components of $\mathbf{z}$ exactly. [6]

(b) Only excitation in the direction of the the eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 1$ will persist (as $|\lambda_2| < 1$ excitation in this direction decays) as $n \to \infty$. Find eingenvectors by solving $A\mathbf{z} = \lambda\mathbf{z}$:
$\underline{\text{for } \lambda_1 = 1}$: bottom equation gives

$$(1-\alpha)x + \tfrac{1}{2}y = y \implies (1-\alpha)x = \tfrac{1}{2}y \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2(1-\alpha) \end{bmatrix},$$

$\underline{\text{for } \lambda_2 = \alpha - \frac{1}{2}}$: top equation gives

$$\alpha x + \tfrac{1}{2}y = (\alpha - \tfrac{1}{2})x \implies \tfrac{1}{2}y = -\tfrac{1}{2}x \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Decomposing $\mathbf{z}_0$ into directions $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{z}_0 = a\mathbf{v}_1 + b\mathbf{v}_2 \implies \begin{array}{c} a+b=1 \\ 2a(1-\alpha)=b \end{array} \implies a = \frac{1}{1+2(1-\alpha)} = \frac{1}{3-2\alpha}.$$

As $n \to \infty$ only the excitation $a\mathbf{v}_1$ persists, so $\mathbf{z}_n \to a\mathbf{v}_1 = \frac{1}{3-2\alpha}[1\ 2(1-\alpha)]^t$. [12]

(c) The only way excitation (unit magnitude of $\mathbf{z}_0$) can decay to zero in a finite number of steps is if $\mathbf{z}_0$ is parallel to $\mathbf{v}_2$ and $\lambda_2 = 0$, therefore $\alpha = \frac{1}{2}$. [12]

Assessors' remarks: This question was a little bit unusual, combining a difference equation with linear algebra. The students generally did not do very well. The overwhelming majority understood how to rewrite the two coupled equations as a single matrix equation. Most students seemed to understand how to use the eigenvalues to examine the iteration scheme, but there was some level of confusion regarding normalisation: most renormalised eigenvectors (which is unnecessary) and, oddly, many renormalised their final answer in (b) to have unit length (which is incorrect). The final question was also not answered that well. Of the candidates who did manage it, about half explicitly examined the twice-forward iteration, and half concluded (more elegantly) that det(A) must be zero.

11. **Linear systems and convolution**

(a) System is second order, as it has two exponential time constants. Impulse response $g(t)$ is derivative of step response, so $g(t) = -Be^{-t} - 2Ce^{-2t}$. Second order systems have continuous impulse responses, so $g(0) = 0 \implies -B - 2C = 0$. Hence $g(t) = 2Ce^{-t} - 2Ce^{-2t} = D(e^{-t} - e^{-2t})$, where $D = 2C$. [10]

(b) We have

$$\begin{aligned} y(t) &= \int_{\tau=0}^{t} g(t-\tau)f(\tau)d\tau \;=\; D\int_{\tau=0}^{t}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)e^{-\tau}d\tau \\ &= De^{-t}\int_{\tau=0}^{t} d\tau - De^{-2t}\int_{\tau=0}^{t} e^{\tau}d\tau \;=\; D(te^{-t} - e^{-2t}[e^t - 1]). \end{aligned}$$

Therefore $y(t) = D(te^{-t} - e^{-t} + e^{-2t})$ for $t > 0$ and $y(t) = 0$ otherwise. [10]

(c) Three intervals. First, for $t < T$, $y(t) = 0$. Second, for $T \le t \le 2T$:

$$\begin{aligned} y(t) &= D\int_{\tau=T}^{t}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)e^{-\tau}d\tau \;=\; De^{-t}\int_{\tau=T}^{t} d\tau - De^{-2t}\int_{\tau=T}^{t} e^{\tau}d\tau \\ &= De^{-t}(t-T) - De^{-2t}\left(e^t - e^T\right), \end{aligned}$$

Finally, for $t > 2T$:

$$\begin{aligned} y(t) &= D\int_{\tau=T}^{2T}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)e^{-\tau}d\tau \;=\; De^{-t}\int_{\tau=T}^{2T} d\tau - De^{-2t}\int_{\tau=T}^{t} e^{\tau}d\tau \\ &= DTe^{-t} - De^{-2t}\left(e^{2T} - e^T\right). \end{aligned}$$

[10]

Assessors' remarks: Part (a): This question was in two parts: How do they know the system is second order and deriving the impulse response. A few candidates made the mistake of saying that the system was second order as it had a discontinuous step response. The derivation of the impulse response was performed correctly by most candidates, although some did not use the correct boundary condition to find D. Part (b): Most candidates got this part correct. Part (c): A large number of candidates made the same error – they swapped the arguments of the convolution, not realising that you cannot do this without adjusting the limits (this works when the limits are $[0, t]$, but here they were integrating over $[T, t]$ or $[T, 2T]$). Candidates who approached the question using linearity and shifting the response to part (b) almost always forgot to scale the response correctly. Many candidates lost marks by not stating the complete response or not stating limits.

12. **Laplace transforms**

(a) For the Laplace transform of $y(t)$ we have

$$\bar{y}(s)(s^2 + 2s + 2) = \frac{6(s+1)}{(s+1)^2 + 1} \implies \bar{y}(s) = \frac{6(s+1)}{((s+1)^2 + 1)^2}$$

[10]

(b) Convolution in the time domain is equivalent to multiplication in the Laplace domain. From (a),

$$\bar{y}(s) = \frac{1}{(s+1)^2 + 1} \times \frac{6(s+1)}{(s+1)^2 + 1} = \bar{g}(s)\bar{f}(s)$$

Hence $y(t) = f(t) * g(t)$, where $g(t) = e^{-t}\sin(t)$ for $t > 0$ and $g(t) = 0$ otherwise, as the inverse Laplace transform of $\bar{g}(s)$. [10]

(c) We take the inverse Laplace transform of the result from (a). Using the fact that $t\sin(t)$ has Laplace transform $2s(s^2+1)^{-2}$ from the *Mathematics Data Book*, and that multiplication by e^{-t} in the time domain corresponds to shifting s by 1 in the Laplace domain, we get $y(t) = 3te^{-t}\sin(t)$ for $t > 0$ and $y(t) = 0$ otherwise. [10]

Assessors' remarks: Part (a): Most candidates got this part correct, barring a few algebraic mistakes. Part (b): Instead of separating the Laplace transform of $y(t)$ and using the inverse Laplace transforms in the databook, many candidates noted that $g(t)$ was the impulse response of the system and therefore found the step response from the differential equation and differentiated it (resulting in a page or two of algebra). This often resulted in answers which had incorrect or unknown constants due to incorrectly applied boundary conditions. Part (c): Three methods were attempted for this part of the question: inspection, convolution, and partial fractions. Those candidates who did this part of the question by inspection mostly got the answer correct. Many candidates noticed that they could find $y(t)$ by solving the convolution integral from part (b). However, this involves a couple of pages of algebra that includes some fiddly trigonometric identities which tripped up many of the candidates. A large number of candidates attempted to solve this part by splitting $\bar{y}(s)$ into partial fractions, none of whom succeeded as this is not a route to a solution for this question. It was clear from comments that many candidates felt frustrated when partial fractions did not yield a result.

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