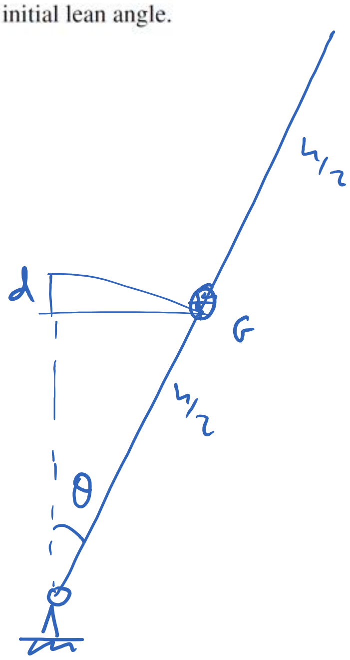
SECTION A

Question 1

1 A famous tower is leaning at an angle of $\theta = 4$ degrees from vertical. It has a total mass m, and can be modelled as a uniform solid cylinder of radius r and height h. At time $t = 0$ the foundations of the tower give way such that it starts to fall, pivoting freely about the centre of the base, which does not move.

(a) Find an expression for the angular velocity of the tower during its fall as a function of the angle of the tower from vertical θ, making a suitable assumption about the effect of the initial lean angle. [8]



$$\Delta PE = \Delta KE$$

$$mgd = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

$$d = \frac{h}{2}(1-\cos\theta) - \underbrace{\overbrace{h/2\left(1-\cos 4\tfrac{\pi}{180}\right)}^{\approx 0.002!\ \text{small so neglect.}}}_{\text{initial PE due to initial angle.}}$$

$$v_G = \omega h/2$$

$$I_G = m\left(\frac{r^2}{4} + \frac{h^2}{12}\right) \quad \text{—— databook.}$$

$$\text{so} \quad \cancel{m}g\frac{h}{2}(1-\cos\theta) = \frac{1}{2}\cancel{m}\,\omega^2 h^2/4 + \frac{1}{2}\cancel{m}\left(\frac{r^2}{4} + \frac{h^2}{12}\right)\omega^2$$

$$gh(1-\cos\theta) = \omega^2 h^2/4 + \omega^2\left(r^2/4 + h^2/12\right)$$

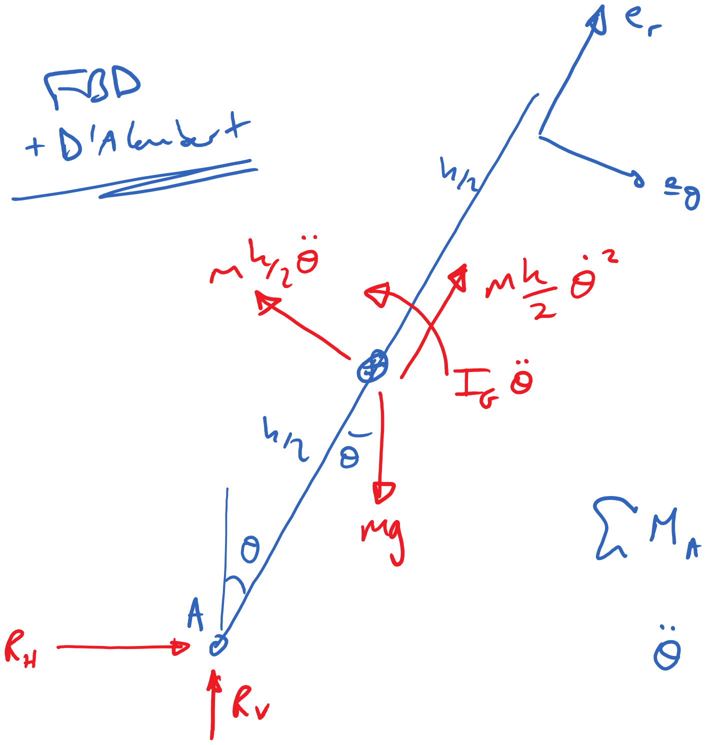
$$\omega^2 = \frac{gh(1-\cos\theta)}{h^2/4 + r^2/4 + h^2/12}$$

$$= \frac{gh(1-\cos\theta)}{r^2/4 + h^2/3}$$

$$\text{so} \quad \omega = \sqrt{\frac{12gh(1-\cos\theta)}{3r^2 + 4h^2}}$$

(b) Find an expression for the angular acceleration of the tower as a function of θ. [5]

FBD + D'Alembert



$$\underline{a} = -r\dot{\theta}^2\underline{e}_r + r\ddot{\theta}\underline{e}_\theta$$

$$\sum M_A \rightarrow mg\frac{h}{2}\sin\theta - m\frac{h}{2}\ddot{\theta}\frac{h}{2} - I_G\ddot{\theta} = 0.$$

$$\ddot{\theta} = \frac{mg\frac{h}{2}\sin\theta}{m\left(\frac{h}{2}\right)^2 + m\left(\frac{r^2}{4} + \frac{h^2}{12}\right)}$$

$$= \frac{6\cancel{m}gh\sin\theta}{3\cancel{m}h^2 + 3\cancel{m}r^2 + \cancel{m}h^2}$$

$$= \frac{6gh\sin\theta}{3r^2 + 4h^2}$$

Alternative method:

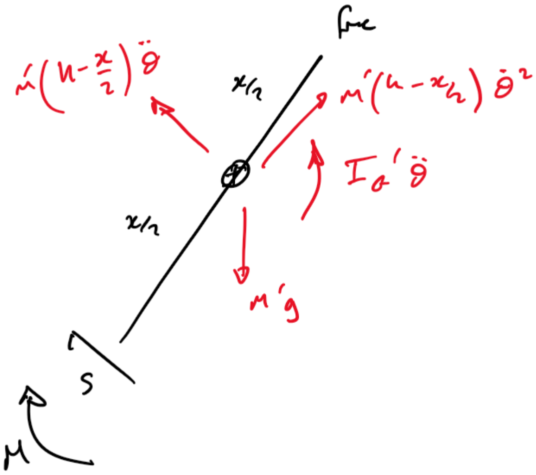
$$\ddot{\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$$

$$= \frac{d}{d\theta}\left(\frac{6gh(1-\cos\theta)}{3r^2+4h^2}\right)$$

$$= \frac{6gh\sin\theta}{3r^2+4h^2}$$

as above.

(c) (i) During the fall, at what distance x, as measured from the top of the tower, does the maximum bending moment occur? [10]



$$\sum M \text{ @ cut}: \quad M = I_G'\ddot{\theta} + m'(h - x/2)\ddot{\theta}\cdot\frac{x}{2} - m'g\frac{x}{2}\sin\theta$$

$$\left(\begin{array}{l} m' = mx/h \\ I_G' = m'\dfrac{x^2}{12} + m'\dfrac{r^2}{4} = \dfrac{mx^3}{12h} + \dfrac{mxr^2}{4h} \end{array} \right)$$

$$M = \left[\frac{mx^3}{12h} + \frac{mxr^2}{4h} + \frac{mx}{h}\left(h - \frac{x}{2}\right)\frac{x}{2}\right]\ddot{\theta} - \frac{mx}{h}g\frac{x}{2}\sin\theta$$

$$= \left[\frac{mx^3}{12h} + \frac{mxr^2}{4h} + \frac{mx^2}{2} - \frac{mx^3}{4h}\right]\ddot{\theta} - \frac{mx^2g}{2h}\sin\theta$$

$$= \left[-\frac{mx^3}{6h} + \frac{mx^2}{2} + \frac{mxr^2}{4h}\right]\ddot{\theta} - \frac{mx^2g}{2h}\sin\theta$$

not functi of x.

$$\frac{dM}{dx} = \left[-\frac{mx^2}{2h} + mx + \frac{mr^2}{4h}\right]\ddot{\theta} - \frac{mxg}{h}\sin\theta = 0$$

@ max.

$$\times\left(\frac{-2h}{m\ddot{\theta}}\right) \Rightarrow \quad x^2 - \frac{2h}{\cancel{m}}\cancel{m}x - \frac{2\cancel{h}}{\cancel{m}}\frac{\cancel{m}r^2}{4\cancel{h}} + \frac{2\cancel{h}}{m\ddot{\theta}}\frac{\cancel{m}xg\sin\theta}{\cancel{h}} = 0.$$

Question 1 continued

$$x^2 + x\underbrace{\left(\frac{2g\sin\theta}{\ddot{\theta}} - 2h\right)}_{\shortparallel} - r^2/2 = 0.$$

$$\frac{2g\cancel{\sin\theta}}{\left(\frac{6gh\cancel{\sin\theta}}{3r^2+4h^2}\right)} = \frac{1}{3}(3r^2+4h^2)/h$$

$$= \frac{r^2}{h} + \frac{4h}{3}$$

$$x^2 + x\left(\frac{r^2}{h} - \frac{2h}{3}\right) - \frac{r^2}{2} = 0.$$

$$x = \frac{\left(\frac{2h}{3} - \frac{r^2}{h}\right) \pm \sqrt{\left(\frac{r^2}{h} - \frac{2h}{3}\right)^2 + 4r^2/2}}{2}$$

$$= \left(h/3 - r^2/2h\right) + \sqrt{\left(h/3 - r^2/2h\right)^2 + r^2/2}$$

('–' solution not physical as –ve).

Note for (c)(i): if you calculate S first and solve S = 0, then you obtain a slightly different answer. The reason for this is that S=dM/dx is only valid for a slender beam.

(ii) Show that $x = 2h/3$ for the case $r = 0$ (i.e. a thin rod). [2]

if $r = 0$, $x = h/3 + \sqrt{(h/3)^2}$

$= 2h/3$ as required.

Question 2

2 Figure 1 shows a diagram (not to scale) for a partially complete design of a lifting mechanism that is made from three light rigid bodies AB, CD and BCE. At the instant shown, the coordinates of the points are A = (0,0) mm, B = (0,60) mm, C = (20,60) mm, D = (x,0) mm and E = (110,90) mm.

(a) The distance x is to be chosen, which determines the length of the link CD.

(i) Identify the coordinates of the instantaneous centre of the completed mechanism such that point E instantaneously moves vertically. [4]

(ii) What is the corresponding distance x that results in E moving vertically? [6]

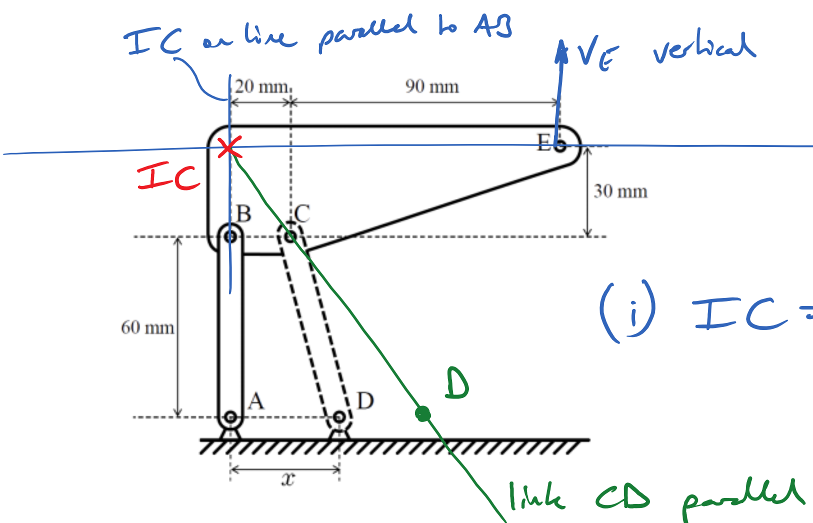
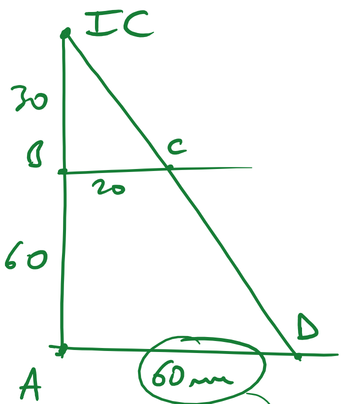


Fig. 1

IC on horizontal line through E

(i) IC = (0, 90) mm



(ii) D = (60, 0) mm

ie x = 60 mm

by similar triangles.

(b) For the case $x = 40\,\text{mm}$:

(i) If point E has an absolute speed of $5\ \text{mm s}^{-1}$ and AB rotates clockwise, find the angular velocity of each component. [10]

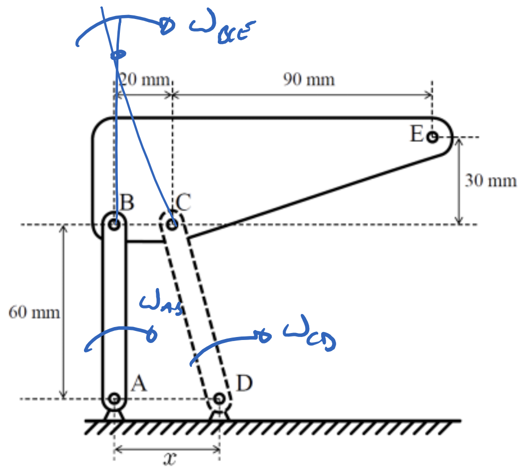
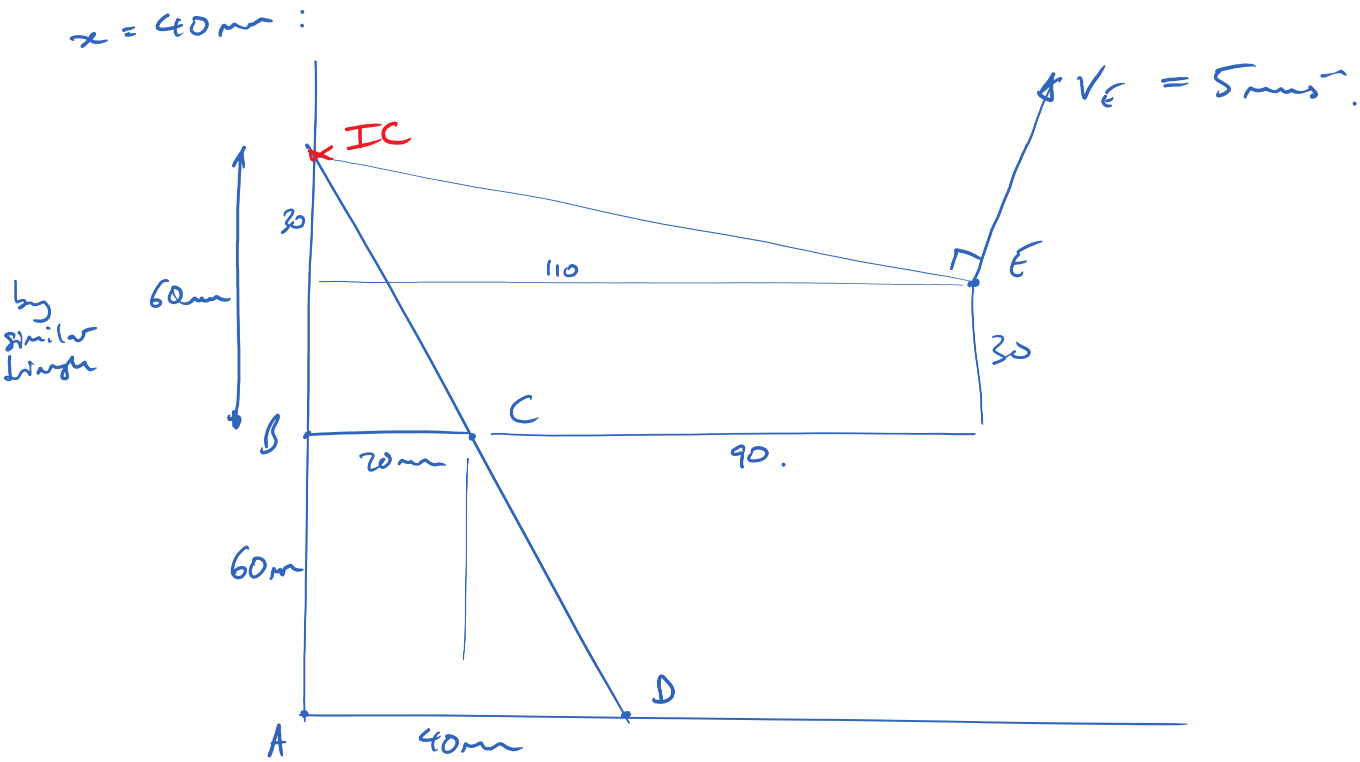


Fig. 1

$x = 40\,\text{mm}$:



$$\omega_{BCE} = \frac{5}{\sqrt{30^2 + 110^2}} = -0.0439\ \text{rad s}^{-1}$$

$$\omega_{AB} = \frac{V_{B/A}}{BA} = \frac{60 \times 0.0439}{60} = +0.0439\ \text{rad s}^{-1}$$

$$\omega_{CD} = \frac{V_{C/D}}{CD} = \frac{\sqrt{60^2+20^2}\cdot 0.0439}{\sqrt{60^2+20^2}} = +0.0439\ \text{rad s}^{-1}$$

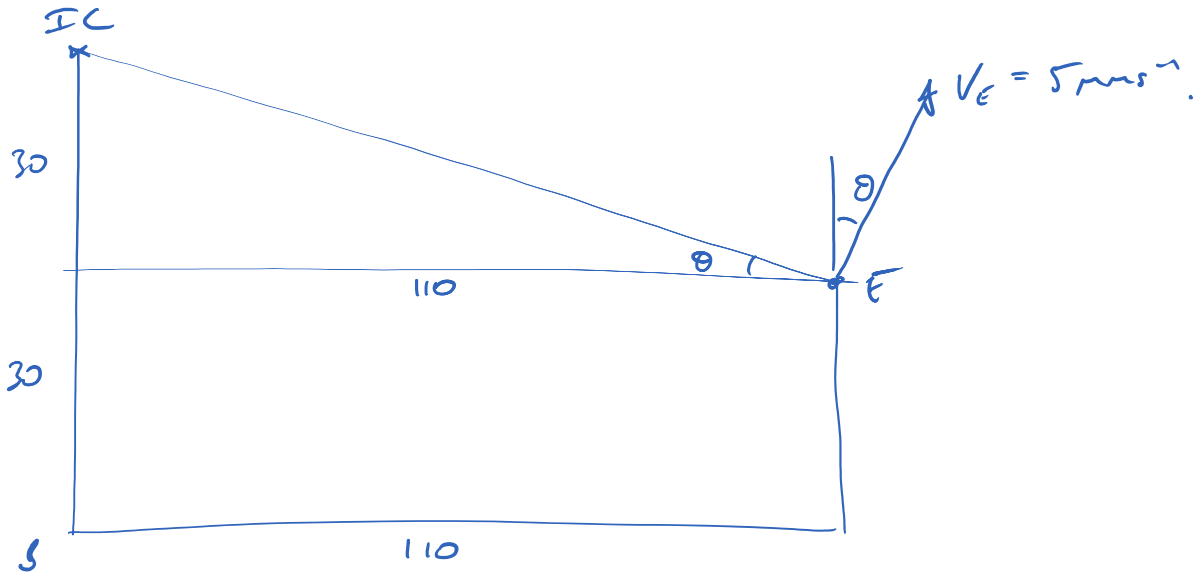
(ii) A frictional torque of 2 N mm opposes the motion at each joint A, B, C and D, and a load of 0.1 N pulls vertically downwards at E. Find the required drive torque at A for the same motion as described in part (b)(i). [5]

$$\omega_A = 0.0439$$

$$\left.\begin{aligned}\omega_B &= 2 \times 0.0439 \\ \omega_C &= 2 \times 0.0439\end{aligned}\right\} \text{opposite directions}$$

$$\omega_D = 0.0439$$

need vertical component of velocity @ E :



$$v\!\uparrow\; = V_E \cos\theta = 5 \cdot \frac{110}{\sqrt{30^2+110^2}} = 4.82\ mms^{-1}$$

by virtual work :

$$T_{input}\,\omega_A = \sum P_{loss}$$

$$= \underbrace{0.0439 \times 6}_{\sum \omega_i \text{ @ each joint}} \times \underbrace{2}_{\text{frict. torque}} + 4.82 \times 0.1$$

$$= 1.01$$

$$T_{input} = \frac{1.01}{0.0439} = 23.0\ Nmm$$

Question 3

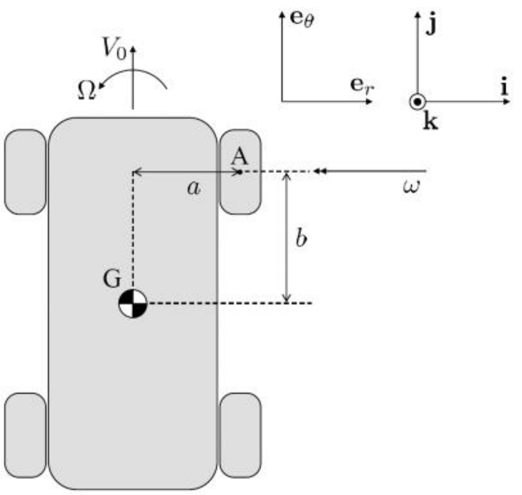
3 A schematic diagram of a go-kart travelling at velocity $V_0\mathbf{j}$ is shown in Fig. 2(a). The centre of mass G is at a height h above the road, and point A is the centre of the front-right wheel. The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed to the ground, and $\mathbf{e_r}, \mathbf{e_\theta}$ are defined relative to the kart, with $\mathbf{e_r}$ parallel to the wheel axes.

Figure 2(b) shows a view from the side of the kart of the front-right wheel with radius r. Point C defines a general position on the tyre at an angle ϕ from the point of contact with the road, and unit vectors $\mathbf{e_1}, \mathbf{e_2}$ rotate with the wheel such that $\mathbf{e_1}$ is in the direction AC. The angular velocity of the kart about a vertical axis is defined to be $\Omega\mathbf{k}$.

(a) Initially, the kart is travelling along a straight path at a constant velocity $V_0\mathbf{j}$ (i.e. $\Omega = 0$) with wheels of radius r rotating at angular speed ω such that there is no slip.

(i) Write down an expression for the position vector of point C relative to G in terms of the unit vectors and parameters defined in Fig. 2. [4]

(ii) Derive expressions for the velocity and acceleration of point C relative to G in terms of the unit vectors and parameters defined in Fig. 2. [6]



(a) Top view of the go-kart

i) $\underline{r}_{C/G} = \underline{r}_{A/G} + \underline{r}_{C/A}$

$= a\underline{e}_r + b\underline{e}_\theta + (r-h)\underline{k} + r\underline{e}_1$

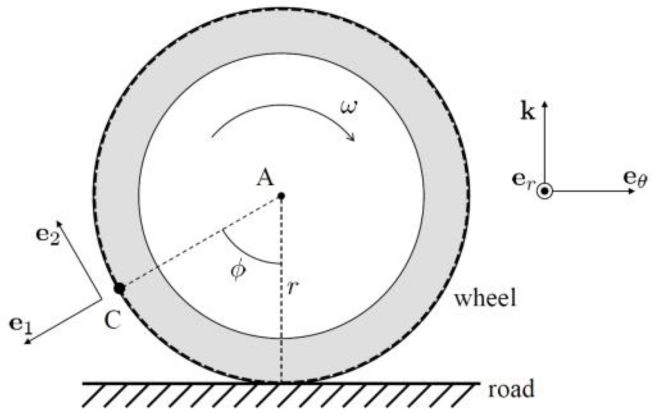
ii) $\dot{\underline{r}}_{C/G} = r\dot{\underline{e}}_1$

$r\omega\underline{e}_2$

[n/b $\underline{e}_r$ & $\underline{e}_\theta$ not rotating]

$\ddot{\underline{r}}_{C/G} = r\omega\dot{\underline{e}}_2$

$= -r\omega^2\underline{e}_1$



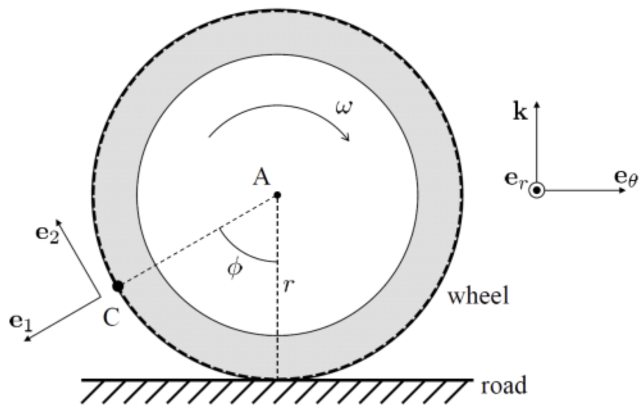
(b) Side view of the front right wheel

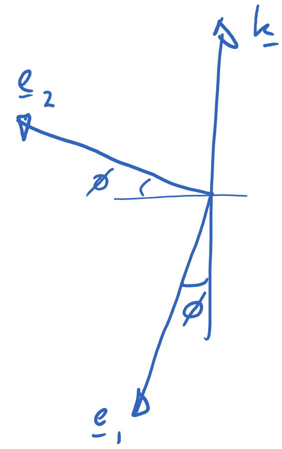
Fig. 2

(b) The kart then goes into a spin about a vertical axis. At the instant shown in Fig. 2(a), the angular velocity of the kart is $\Omega\mathbf{k}$ where Ω is constant, the wheels continue to rotate at constant angular speed ω, and the centre of mass of the kart continues to move at constant velocity $V_0\mathbf{j}$. It can be assumed that the driver does not steer the wheels.

(i) Derive expressions for the derivatives $\dot{\mathbf{e}}_r$, $\dot{\mathbf{e}}_\theta$, $\dot{\mathbf{e}}_1$ and $\dot{\mathbf{e}}_2$. [6]

(ii) When C is instantaneously in contact with the ground, find the velocity and acceleration of point C relative to the ground in terms of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and $\mathbf{e}_r, \mathbf{e}_\theta$, and the parameters defined in Fig. 2. [9]





(b) Side view of the front right wheel

Fig. 2

i) (e_r, e_θ) rotating @ (Ωk), (e_1, e_2) rotating @ $(\Omega k - \omega e_r)$

$\dot{e}_r = (\Omega k \times e_r) = \Omega e_\theta$ //

$\dot{e}_\theta = (\Omega k \times e_\theta) = -\Omega e_r$ //

$\dot{e}_1 = (\Omega k - \omega e_r) \times e_1$
$= \Omega \sin\phi \, e_r + \omega e_2$ //

$\dot{e}_2 = (\Omega k - \omega e_r) \times e_2$
$= \Omega \cos\phi \, e_r - \omega e_1$ //

ii) $r_c = r_G + a e_r + b e_\theta + (r-h)k + r e_1$

$\dot{r}_c = V_0 j + a\dot{e}_r + b\dot{e}_\theta + r\dot{e}_1$

$= V_0 j + a\Omega e_\theta - b\Omega e_r + r(\Omega \sin\phi \, e_r + \omega e_2)$ —— general ϕ

$\dot{r}_c|_{\phi=0} = V_0 j + a\Omega e_\theta - b\Omega e_r + r\omega e_2$

$= V_0 j + (a\Omega - r\omega)e_\theta - b\Omega e_r$ // —— $\phi = 0$.

(as $e_2 = -e_\theta$ instantaneously)

$$\dot{\underline{r}}_c = V_0 \underline{j} + a\Omega \underline{e}_\theta - b\Omega \underline{e}_r + r(\Omega \sin\phi\, \underline{e}_r + \omega \underline{e}_2)$$

need general expression to differentiate correctly

$$\ddot{\underline{r}}_c = a\Omega \dot{\underline{e}}_\theta - b\Omega \dot{\underline{e}}_r + r\Omega\omega\cos\phi\, \underline{e}_r + r\Omega\sin\phi\, \dot{\underline{e}}_r + r\omega \dot{\underline{e}}_2$$

@ $\phi = 0 \Rightarrow$

$$\ddot{\underline{r}}_c = \underline{-a\Omega^2 \underline{e}_r} - b\Omega^2 \underline{e}_\theta + \underline{r\Omega\omega \underline{e}_r} + \underline{r\omega\Omega \underline{e}_r} - r\omega^2 \underline{e}_1$$

$= -\underline{k}$ instantaneously

$$= (-a\Omega^2 + 2r\omega\Omega)\underline{e}_r - b\Omega^2 \underline{e}_\theta + r\omega^2 \underline{k}$$

Note there are several ways to attempt this question. If you first set $\phi = 0$, then extra care must be taken:

$$\underline{r}_c = a\underline{e}_r + b\underline{e}_\theta + (r-h)\underline{k} + r\underline{e}_1$$

$$\dot{\underline{r}}_c = a\Omega \underline{e}_\theta - b\Omega \underline{e}_r + r(\Omega \underline{k} - \omega \underline{e}_r) \times \underline{e}_1$$

$$= a\Omega \underline{e}_\theta - b\Omega \underline{e}_r + r\omega \underline{e}_2$$

$\underline{e}_2 = -\underline{e}_\theta$

must use this expr otherwise miss terms.

$$\ddot{\underline{r}}_c = -a\Omega^2 \underline{e}_r - b\Omega^2 \underline{e}_\theta + r(-\omega \dot{\underline{e}}_r) \times \underline{e}_1 + r(\Omega \underline{k} - \omega \underline{e}_r) \times (\Omega \underline{k} - \omega \underline{e}_r) \times \underline{e}_1$$

$$= -a\Omega^2 \underline{e}_r - b\Omega^2 \underline{e}_\theta - r\omega\Omega\, \underline{e}_\theta \times \underline{e}_1 + r(\Omega \underline{k} - \omega \underline{e}_r) \times (\omega \underline{e}_2)$$

$$= -a\Omega^2 \underline{e}_r - b\Omega^2 \underline{e}_\theta + r\omega\Omega\, \underline{e}_r + r\omega\Omega \underline{e}_r - r\omega^2 \underline{e}_1$$

$\underline{e}_1 = -\underline{k}$

$$= (-a\Omega^2 + 2r\omega\Omega)\underline{e}_r - b\Omega^2 \underline{e}_\theta + r\omega^2 \underline{k}$$

as before.

A non-exhaustive list of accepted variations:

(a) (i) $$\underline{r}_{c/G} = a\underset{\underline{e}_r}{\underset{\shortparallel}{\underline{i}}} + b\underline{j} - r\sin\phi\,\underset{\underline{e}_\theta}{\underset{\shortparallel}{\underline{j}}} - \left(h - r + r\cos\phi\right)\underline{k}$$

$$= a\underline{e}_r + b\underline{e}_\theta + (r-h)\underline{k} + r\underline{e}_1$$

(ii) $$\dot{\underline{r}}_{c/G} = r\omega\underline{e}_2 = V_0\underline{e}_2$$

$$= -r\omega\cos\phi\,\underset{\underline{e}_\theta}{\underset{\shortparallel}{\underline{j}}} + r\omega\sin\phi\,\underline{k}$$

$$\ddot{\underline{r}}_{c/G} = -r\omega^2\underline{e}_1 = -\frac{V_0^2}{r}\underline{e}_1$$

$$= r\omega^2\sin\phi\,\underset{\underline{e}_\theta}{\underset{\shortparallel}{\underline{j}}} + r\omega^2\cos\phi\,\underline{k}$$

(b) (i) $$\dot{\underline{e}}_r = \Omega\underline{e}_\theta = \Omega\underline{j}$$

$$\dot{\underline{e}}_\theta = -\Omega\underline{e}_r = -\Omega\underline{i}$$

$$\dot{\underline{e}}_1 = \Omega\sin\phi\,\underline{e}_r + \omega\underline{e}_2$$

$$= \Omega\sin\phi\,\underline{i} - \omega\cos\phi\,\underset{\underline{j}}{\underset{\shortparallel}{\underline{e}_\theta}} + \omega\sin\phi\,\underline{k}$$

$$\dot{\underline{e}}_2 = \Omega\cos\phi\,\underline{e}_r - \omega\underline{e}_1$$

$$= \Omega\cos\phi\,\underline{i} + \omega\sin\phi\,\underline{e}_\theta + \omega\cos\phi\,\underline{k}$$

(ii) $$\dot{\underline{r}}_c = \left(V_0 + a\Omega - r\omega\right)\underset{\underline{j}}{\underset{\shortparallel}{\underline{e}_\theta}} - b\Omega\underline{e}_r$$

$\underline{e}_\theta \neq -\underline{e}_2$

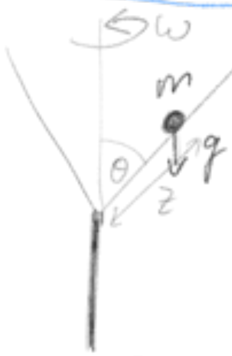
but qn asks for expression in terms of ($i\,j\,k\,e_r e_\theta$)

($= a\Omega\underline{e}_\theta - b\Omega\underline{e}_r$ instantaneously as $V_0 = r\omega$ from (a) and part (b) says kart continues at same ω and V_0.)

$$\ddot{\underline{r}}_c = \left(2r\omega\Omega - a\Omega^2\right)\underset{\underline{i}}{\underset{\shortparallel}{\underline{e}_r}} - b\Omega^2\underset{\underline{j}}{\underset{\shortparallel}{\underline{e}_\theta}} + r\omega^2\underline{k}$$

SECTION B

Q4 olive in martini glass



a) $KE = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mz^2\omega^2\sin^2\theta$

$PE = mgz\cos\theta$ 3

$$\boxed{L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mz^2\omega^2\sin^2\theta - mgz\cos\theta}$$

EofM $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$ $q \equiv z$

$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$ $\frac{\partial L}{\partial z} = mz\omega^2\sin^2\theta - mg\cos\theta$

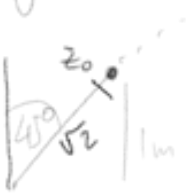
$$\boxed{\ddot{z} = z\sin^2\theta\,\omega^2 - g\cos\theta}$$ 5

b) equilibrium : $\ddot{z} = 0 \Rightarrow \boxed{z_0 = \frac{g\cos\theta}{\omega^2\sin^2\theta}}$ 2

increasing z away from eqm : $\ddot{z} > 0$ 2

$\Rightarrow$ unstable equilibrium

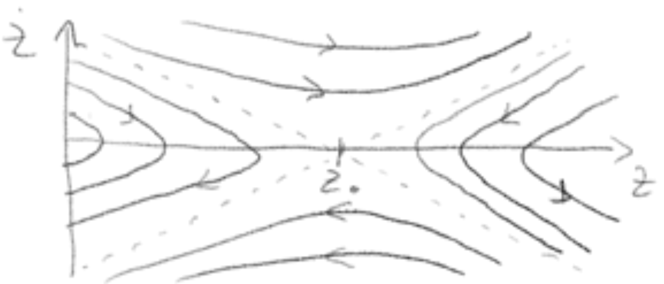
$g = 9.8$ $\omega = \sqrt{10}$ $\theta = \pi/2 \Rightarrow z_0 = \sqrt{2}\,\frac{9.8}{10} < \ell$

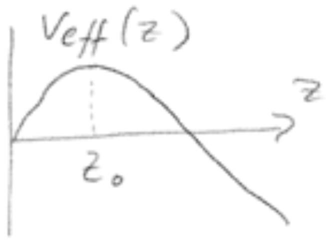


$\Rightarrow$ $\boxed{\text{bead flies off rod}}$ 3

c) $\dot{z}\ddot{z} = \dot{z}z^2\omega^2\sin^2\theta - \dot{z}g\cos\theta \Rightarrow \dot{z}^2 - z^2\omega^2\sin^2\theta + zg\cos\theta = \text{const}$ 2

locus is a pair of hyperbolas

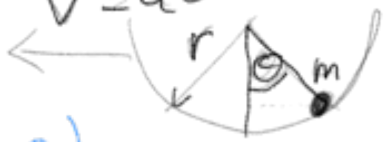




8

Q5 egg and spoon race

$v = at$



$$T = \frac{1}{2}m(r\dot{\theta}\cos\theta - at)^2 + \frac{1}{2}mr^2\dot{\theta}^2\sin^2\theta$$

a)

$$V = rmg(1-\cos\theta)$$

6

$$\boxed{L = T - V = \frac{1}{2}m(r\dot{\theta}\cos\theta - at)^2 + \frac{1}{2}mr^2\dot{\theta}^2\sin^2\theta + rmg\cos\theta}$$

b)

$$\frac{\partial L}{\partial \theta} = -mgr\sin\theta + mr^2\dot{\theta}^2\cos\theta\sin\theta - m(r\dot{\theta}\cos\theta - at)r\dot{\theta}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m(r\dot{\theta}\cos\theta - at)r\cos\theta + mr^2\dot{\theta}\sin^2\theta$$

$$= mart\cos\theta + mr^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mar\cos\theta + mr^2\ddot{\theta} + mart\sin\theta\,\dot{\theta}$$

$$\Rightarrow mr\ddot{\theta} + mar\cos\theta + mart\dot{\theta}\sin\theta = -mgr\sin\theta + mr^2\dot{\theta}^2\sin\theta\cos\theta - \quad + mart\dot{\theta}\sin\theta - mr^2\dot{\theta}^2\sin\theta\cos\theta$$

$$\boxed{\ddot{\theta} = -\frac{1}{r}(g\sin\theta - a\cos\theta) \qquad \ddot{\theta} = -\frac{\sqrt{g^2+a^2}}{r}\sin(\theta+\beta), \quad \tan\beta = \frac{a}{g}}$$

7

no explicit time dependence! Pendulum equation around stable angle β (not SHM)

6

c) stable equilibrium possible: $\beta < \varphi/2$

$$\frac{a}{g} < \tan\frac{\varphi}{2} \Rightarrow \boxed{a < g\tan\varphi/2}$$

6

Q6 double pendulum

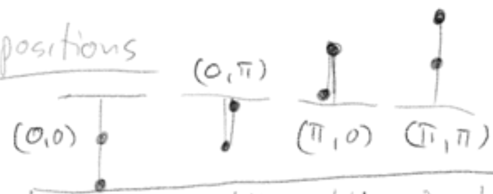
a) $T = \frac{1}{2}m\left[\ell^2\dot{\theta}_1^2 + (\ell\dot{\theta}_1\cos\theta_1 + \ell\dot{\theta}_2\cos\theta_2)^2 + (\ell\dot{\theta}_1\sin\theta_1 + \ell\dot{\theta}_2\sin\theta_2)^2\right]$

$V = -mg\ell\cos\theta_1 - mg\ell(\cos\theta_1 + \cos\theta_2)$

$$\boxed{L = \frac{1}{2}m\ell^2(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)) + mg\ell(2\cos\theta_1 + \cos\theta_2)}$$ 6

b)

egm positions



2

2 stable modes	1 stable 1 unstable	1 stable 1 unstable	2 unstable modes

4

Q: stability of modes not of egm. states

(Two lowest energy equilibria only also accepted)

c) second lowest PE: $\theta_1 = 0$, $\theta_2 = \pi$, expand L

$$L = \frac{1}{2}m\ell^2\left[2\delta\dot{\theta}_1^2 + \delta\dot{\theta}_2^2 + 2\delta\dot{\theta}_1\delta\dot{\theta}_2\cos(\delta\theta_1 - \pi - \delta\theta_2)\right] + mg\ell\left[2\left(1 - \frac{\delta\theta_1^2}{2}\right) + (-1) + \frac{\delta\theta_2^2}{2}\right]$$

$$= \frac{1}{2}m\ell^2[\delta\dot{\theta}_1, \delta\dot{\theta}_2]\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}\delta\dot{\theta}_1\\ \delta\dot{\theta}_2\end{bmatrix} + \frac{1}{2}mg\ell[\delta\theta_1, \delta\theta_2]\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\delta\theta_1\\ \delta\theta_2\end{bmatrix} + \text{const}$$ 4

d) normal modes:

$$K = -mg\ell\begin{bmatrix}-2 & 0\\ 0 & 1\end{bmatrix} \qquad M = m\ell^2\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$$

$$\det(K - \omega^2 M) = 0$$

$$\begin{vmatrix}2\omega^2 - 2g/\ell & -\omega^2\\ -\omega^2 & \omega^2 + g/\ell\end{vmatrix} = 0$$

$$2(\omega^2 - g/\ell)(\omega^2 + g/\ell) - \omega^4 = 0$$

$$\omega^4 - 2g^2/\ell^2 = 0$$ 6

$$\omega^4 = 2g^2/\ell^2 \Rightarrow \boxed{\omega^2 = \pm\sqrt{2}\,g/\ell}$$ 3

real root: stable mode, imaginary root: unstable mode