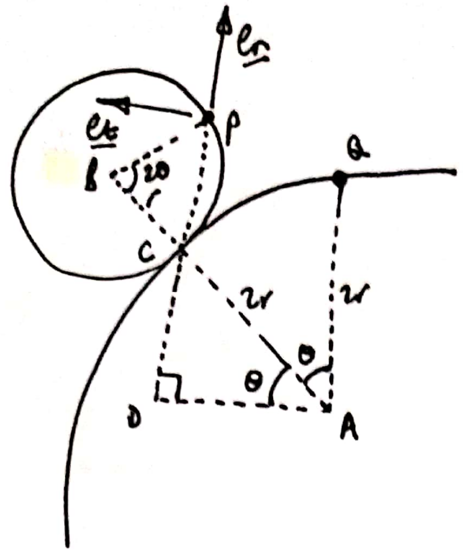
1a)



Rolling ∴ arc CP = CQ arc length

∴ $\underline{CP} = 2r\theta\sin\theta\, \underline{e_n}$

$\underline{AP} = \underline{AD} + \underline{DP} = 2r\cos\theta\,\underline{e_t} + 2r\sin\theta\,\underline{e_n} + \underline{CP}$

$= 2r\cos\theta\,\underline{e_t} + 4r\sin\theta\,\underline{e_n}$

$\underline{v_P} = -2r\dot\theta\sin\theta\,\underline{e_t} + 2r\cos\theta\,\underline{\dot e_t}$

$+ 4r\dot\theta\cos\theta\,\underline{e_n} + 4r\sin\theta\,\underline{\dot e_n}$

$\underline{\dot e} = \underline{\omega} \times \underline{e}$ $\quad \omega$ for unit vector $= 2\dot\theta$

$= \omega\underline{k} \times \underline{e}$

∴ $\underline{\dot e_t} = 2\dot\theta\,\underline{k} \times \underline{e_t} = -2\dot\theta\,\underline{e_n}$; $\underline{\dot e_n} = 2\dot\theta\,\underline{k} \times \underline{e_n} = 2\dot\theta\,\underline{e_t}$

∴ $\underline{v_P} = -2r\dot\theta\sin\theta\,\underline{e_t} - 4r\dot\theta\cos\theta\,\underline{e_n} + 4r\dot\theta\cos\theta\,\underline{e_n} + 8r\dot\theta\sin\theta\,\underline{e_t}$

$= 6r\dot\theta\sin\theta\,\underline{e_t}$

b) $\underline{\dot v_P} = 6r\ddot\theta\sin\theta\,\underline{e_t} + 6r\dot\theta^2\cos\theta\,\underline{e_t} + 6r\dot\theta\sin\theta\,\underline{\dot e_t}$

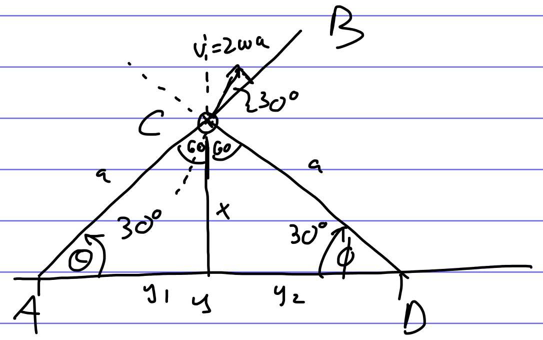
$= \left(6r\ddot\theta\sin\theta + 6r\dot\theta^2\cos\theta\right)\underline{e_t} - 12r\dot\theta^2\sin\theta\,\underline{e_n}$

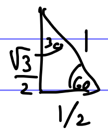
c) $\underline{e_n}$ component of accn. $= (\text{speed}^2)/\rho$

∴ $12r\dot\theta^2\sin\theta = \dfrac{(6r\dot\theta\sin\theta)^2}{\rho}$

∴ $\rho = 3r\sin\theta$

a)





$a \sin\phi = x$

$a\cos\phi = y_2$

$y_1 \tan\theta = x = a\sin\phi$

$\cos 30^\circ = \sqrt{1 - 1/4} = \frac{\sqrt{3}}{2}$

$\sin 30^\circ = \frac{1}{2}$

$\tan 30^\circ = 1/\sqrt{3}$

$$y = y_1 + y_2 = a\cos\phi + \frac{a\sin\phi}{\tan\theta} = \text{const}$$

$$\frac{d}{dt}: \quad -\sin\phi\,\dot\phi + \frac{\cos\phi}{\tan\theta}\dot\phi - \frac{\sin\phi}{(\tan\theta)^2}\sec^2\theta\,\dot\theta = 0$$

$$-\tfrac{1}{2}\dot\phi + \frac{\sqrt{3}}{2}\sqrt{3}\,\dot\phi - \tfrac{1}{2}\,3\,\tfrac{4}{3}\,\dot\theta = 0$$

$$-\dot\phi + 3\dot\phi - 4\dot\theta = 0$$

$$2\dot\phi = 4\dot\theta$$

$$\dot\phi = 2\dot\theta \boxed{= 2w}$$

-ve due to def. of ϕ

speed of C $2wa$, projected along AB rod:

$$2wa\cos 30^\circ = \boxed{\sqrt{3}\,wa}$$

b) Virtual power:

$$-F\cdot v + Tw = 0$$

$$-F\sqrt{3}wa + Tw = 0$$

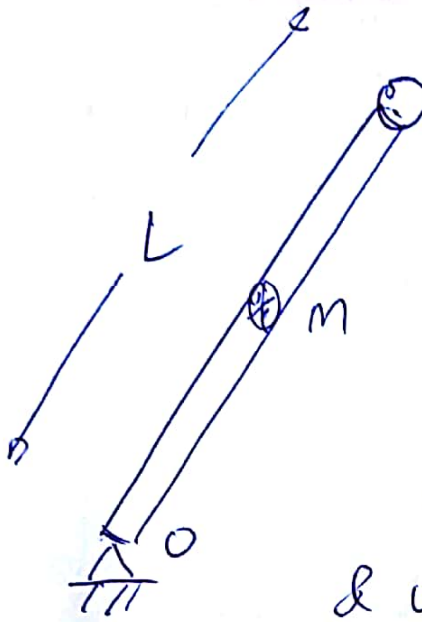
$$T = \sqrt{3}aF$$

c) pin C is rotating with angular velocity $\underline{3w}$

$$0 = -F\cdot v + Tw - \underset{A}{Qw} - \underset{D}{Q2w} - \underset{C}{Q3w}$$

$$0 = -\sqrt{3}aF + T - 6Q$$

$$\boxed{T = 6Q + \sqrt{3}aF}$$



a/ $I_0 = \frac{1}{3}ML^2 + ML^2$

$= \frac{4}{3}ML^2$

$PE + KE = \text{const}$

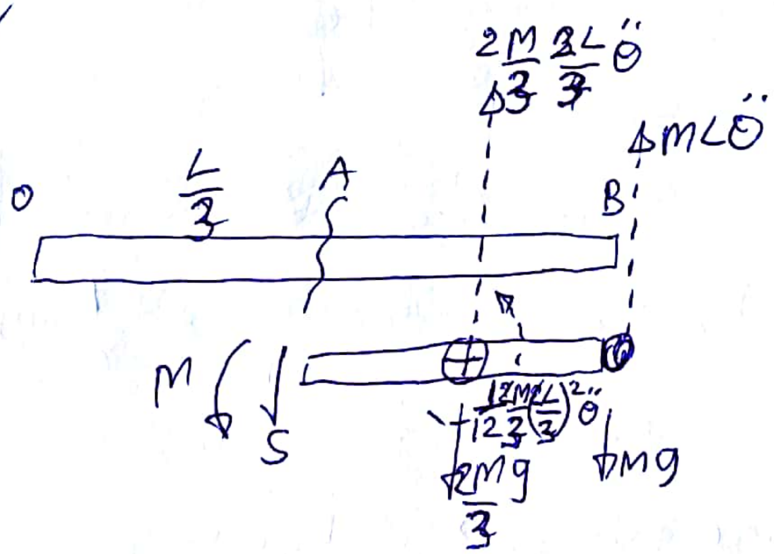
$\therefore \frac{1}{2} I_0 \dot{\theta}^2 = Mg\left(\frac{L}{2} + L\right)(1 - \cos\theta)$

& use $\ddot{\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$

$\therefore I_0 \ddot{\theta} = Mg \frac{3L}{2} \sin\theta$

$\therefore \boxed{\ddot{\theta} = \frac{9g}{8L} \sin\theta}$

b/



Cut at A

$\ddot{\theta} = \frac{9g}{8L}$

$S = \left(\frac{2M}{3}\frac{2L}{3} + ML\right)\ddot{\theta} - \frac{5mg}{3}$

$= mg\left[\frac{4L}{9} + L\right]\frac{9}{8}\frac{9}{L} - \frac{5}{3}mg$

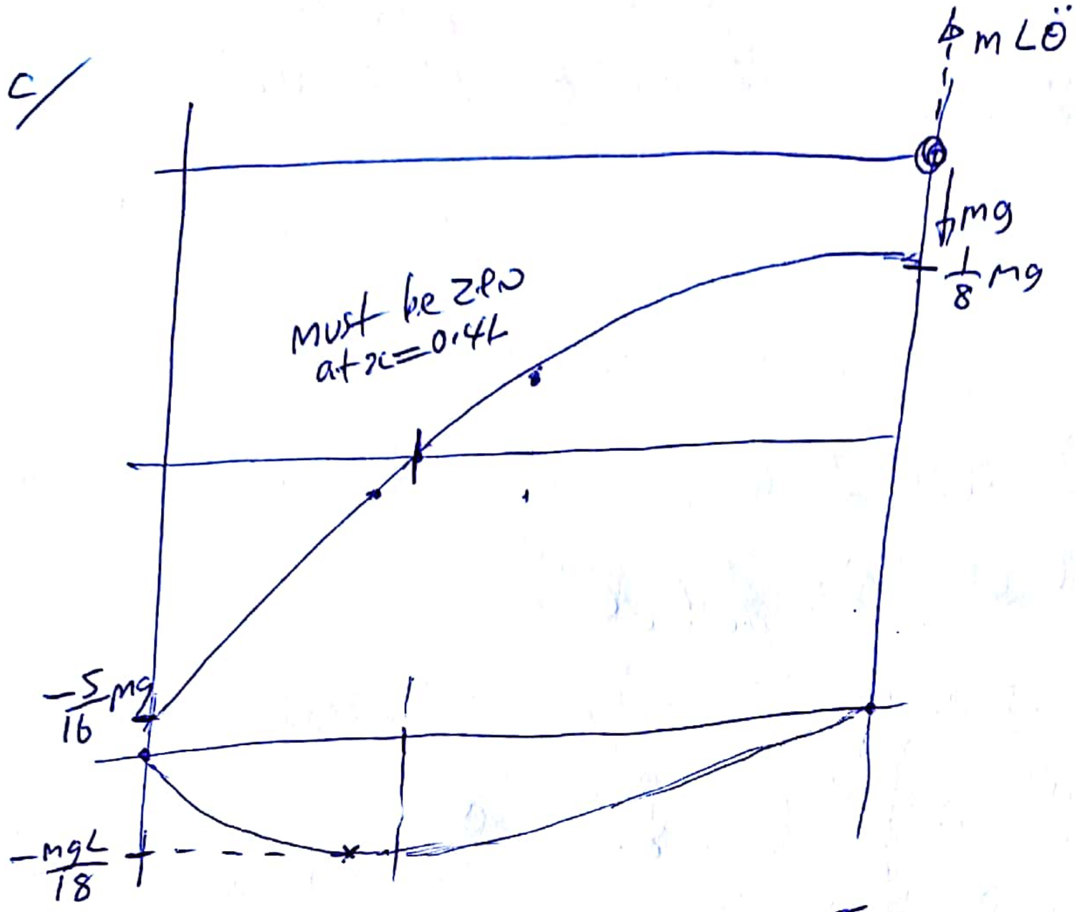
$= mg\left[\frac{13}{9}\frac{9}{8} - \frac{5}{3}\right] = \cancel{\frac{3mg}{64}} \quad -\frac{mg}{24}$

$M = \frac{2mg}{3}\frac{L}{3} + mg\frac{2L}{3} - \ddot{\theta}\left(\frac{2M}{3}\frac{2L}{3}\frac{L}{3} + ML\frac{2L}{3} + \frac{1}{12}\frac{2m}{3}\left(\frac{2L}{3}\right)^2\right)$

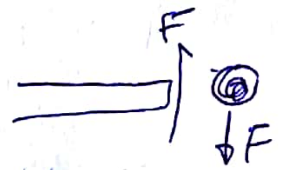
$= mgL\left(\frac{2}{9} + \frac{2}{3} - \frac{9}{8}\left(\frac{4}{27} + \frac{2}{3} + \frac{2}{81}\right)\right) = -\frac{mgL}{18}$

Scanned by CamScanner

c/



BMD

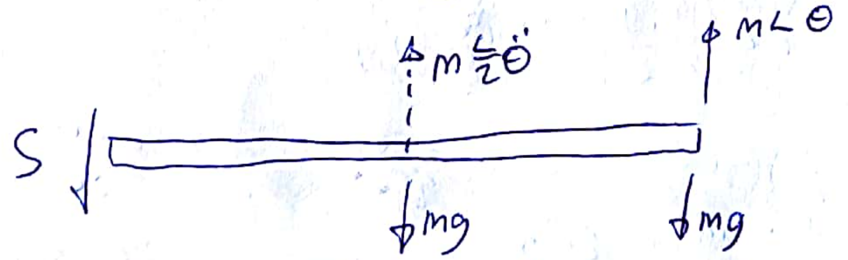


Shear force at B

$$F = mL\ddot{\theta} - mg = \frac{1}{8}mg = S$$

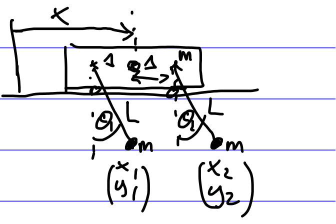
shear force at O

$= M'$



$$S = \frac{3m}{2}L\ddot{\theta} - 2mg = \left(\frac{3}{2}\frac{9}{8} - 2\right)mg = -\frac{5}{16}mg$$

(4)



$$x_1 = x - \Delta + L\sin\theta_1$$
$$y_1 = L\cos\theta_1$$

$$x_2 = x + \Delta + L\sin\theta_2$$
$$y_2 = L\cos\theta_2$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right)$$
$$= \tfrac{1}{2}m\left(\dot{x}^2 + (\dot{x} + L\cos\theta_1\dot{\theta}_1)^2 + L^2\sin^2\theta_1\dot{\theta}_1^2 + (\dot{x} + L\cos\theta_2\dot{\theta}_2)^2 + L^2\sin^2\theta_2\dot{\theta}_2^2\right)$$
$$V = Lmg\left(1 - \cos\theta_1 + 1 - \cos\theta_2\right)$$

$$L = T - V$$
$$= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{x}^2 + 2L\dot{x}\cos\theta_1\dot{\theta}_1 + L^2\dot{\theta}_1^2 + \dot{x}^2 + 2L\dot{x}\cos\theta_2\dot{\theta}_2 + L^2\dot{\theta}_2^2\right) - Lmg\left(2 - \cos\theta_1 - \cos\theta_2\right)$$

$$= \tfrac{3}{2}m\dot{x}^2 + \tfrac{1}{2}mL^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + Lm\dot{x}(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2) + Lmg(\cos\theta_1 + \cos\theta_2 - 2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0$$

$$0 = 3m\ddot{x} + Lm\left(\cos\theta_1\ddot{\theta}_1 - \sin\theta_1\dot{\theta}_1^2 + \cos\theta_2\ddot{\theta}_2 - \sin\theta_2\dot{\theta}_2^2\right)$$

$$0 = L^2m\ddot{\theta}_1 + Lm(\ddot{x}\cos\theta_1 - \dot{x}\sin\theta_1\dot{\theta}_1) + Lm(\dot{x}\dot{\theta}_1 + g)\sin\theta_1$$

$$0 = L^2m\ddot{\theta}_2 + Lm(\ddot{x}\cos\theta_2 - \dot{x}\sin\theta_2\dot{\theta}_2) + Lm(\dot{x}\dot{\theta}_1 + g)\sin\theta_2$$

Linearize: small θ_1, θ_2

$$0 \approx 3m\ddot{x} + Lm(\ddot{\theta}_1 + \ddot{\theta}_2)$$
$$0 \approx L^2m\ddot{\theta}_1 + Lm\ddot{x} + Lmg\theta_1$$
$$0 \approx L^2m\ddot{\theta}_2 + Lm\ddot{x} + Lmg\theta_2$$

$$\begin{bmatrix} 3m & Lm & Lm \\ Lm & L^2m & 0 \\ Lm & 0 & L^2m \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & Lmg & 0 \\ 0 & 0 & Lmg \end{bmatrix}\begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} = 0$$

trial solution: $x = Ae^{i\omega t}$; $\theta_1 = B_1e^{i\omega t}$; $\theta_2 = B_2e^{i\omega t}$

$$\begin{bmatrix} -3m\omega^2 & -Lm\omega^2 & -Lm\omega^2 \\ -Lm\omega^2 & Lm(g - L\omega^2) & 0 \\ -Lm\omega^2 & 0 & Lm(g - L\omega^2) \end{bmatrix}\begin{bmatrix} A \\ B_1 \\ B_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -3/L & -1 & -1 \\ -1 & g/\omega^2 - L & 0 \\ -1 & 0 & g/\omega^2 - L \end{bmatrix}\begin{bmatrix} A \\ B_1 \\ B_2 \end{bmatrix} = 0$$

(1) $\omega^2 = 0 \Rightarrow A = 1,\ B_1 = B_2 = 0$

(2) $\omega^2 = g/L \Rightarrow A = 0,\ B_1 = -B_2$

(3) $\omega^2 = \frac{g}{L}\frac{1}{1 - 2/3} = \frac{3g}{L} \qquad A = 1,\ B_1 = B_2 = -\frac{3}{2L}$

(1) free rolling
(2) stationary trolley, antiphase pendulum
(3) inphase pendulums, oscillating trolley

5 a) $T = \frac{1}{2}m\left(\omega^2(R + l\sin\theta)^2 + (l\dot{\theta})^2\right)$

$T = \frac{1}{2}m\left(\omega^2(R^2 + 2Rl\sin\theta + l^2\sin^2\theta) + l^2\dot{\theta}^2\right)$

$V = mgl(1-\cos\theta)$

b) $L = T - V$

$= \frac{1}{2}m\left(\omega^2(R^2 + 2Rl\sin\theta + l^2\sin^2\theta) + l^2\dot{\theta}^2\right) - mgl(1-\cos\theta)$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ml^2 2\dot{\theta} = ml^2\dot{\theta}$

$F_\theta = \frac{\partial L}{\partial \theta} = \frac{1}{2}m\omega^2(2Rl\cos\theta + l^2 2\sin\theta\cos\theta) - mgl\sin\theta$

$\frac{d}{dt}(p_\theta) = F_\theta \therefore ml^2\ddot{\theta} = \frac{1}{2}m\omega^2(2Rl\cos\theta + l^2 2\sin\theta\cos\theta)$
$- mgl\sin\theta$

$l\ddot{\theta} = \omega^2 R\cos\theta + l\sin\theta\cos\theta - g\sin\theta$

$\ddot{\theta} - \omega^2\cos\theta\left(\frac{R}{l} + \sin\theta\right) + \frac{g}{l}\sin\theta = 0.$

c) equilibrium when $\ddot{\theta} = \dot{\theta} = 0$

$\omega^2\left(\frac{R}{l}\cos\theta + \sin\theta\cos\theta\right) = \frac{g}{l}\sin\theta.$

$\div\cos\theta$ $\qquad \omega^2\left(\frac{R}{l} + \sin\theta\right) = \frac{g}{l}\tan\theta.$

$\frac{R}{l} + \sin\theta = \frac{g}{\omega^2 l}\tan\theta$

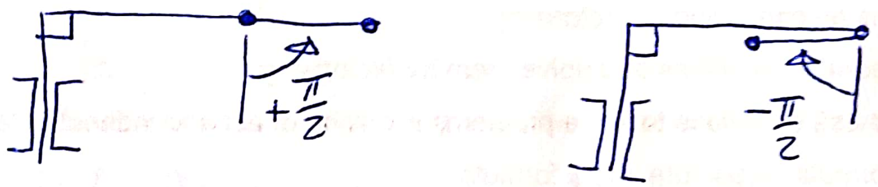
d)

$$\frac{R}{l} + \sin\theta = \frac{g}{\omega^2 l}\tan\theta .$$

now $R \gg l$, $\omega^2 \gg \frac{g}{l}$.

$$\therefore \quad +\infty = \tan\theta_0$$

$$\underline{\underline{\theta_0 = \pm\frac{\pi}{2}}}$$



EoM from part (b) :

$$\ddot{\theta} - \omega^2\cos\theta\left(\frac{R}{l} + \sin\theta\right) + \frac{g}{l}\sin\theta = 0.$$

$\frac{R}{l} \gg 1 \quad \therefore \quad \ddot{\theta} \simeq \frac{R}{l}\omega^2\cos\theta + \frac{g}{l}\sin\theta$

$\omega^2 \gg \frac{g}{l} \quad \therefore \quad \ddot{\theta} \simeq \frac{R}{l}\omega^2\cos\theta.$

let $\theta = \theta_0 + \delta\theta \Rightarrow \ddot{\theta} = \ddot{\delta\theta}$

$$\ddot{\delta\theta} = \frac{R}{l}\omega^2\cos(\theta_0 + \delta\theta)$$

$$\left[\begin{array}{l}\cos(\theta_0 + \delta\theta) = \cos\theta_0\cos\delta\theta - \sin\theta_0\sin\delta\theta . \\ \delta\theta \text{ small } \therefore \quad = \cos\theta_0 - \delta\theta\sin\theta_0\end{array}\right]$$

hence. $\ddot{\delta\theta} = \frac{R}{l}\omega^2(\cos\theta_0 - \delta\theta\sin\theta_0)$

$\theta_0 = +\frac{\pi}{2} \qquad \ddot{\delta\theta} = \frac{R\omega^2}{l}(0 - \delta\theta) \quad \therefore$ stable

$\theta_0 = -\frac{\pi}{2} \qquad \ddot{\delta\theta} = \frac{R\omega^2}{l}(0 + \delta\theta) \quad \therefore$ unstable. $\|$

6 a) $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2$

$V = \frac{1}{2}kr^2$

$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}kr^2$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mr^2 2\dot{\theta} = mr^2\dot{\theta}$

$F_\theta = \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}(p_\theta) = F_\theta = 0 \quad \therefore \quad p_\theta = mr^2\dot{\theta} = h$ (constant)

b) $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad F_r = \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - kr$

EoM for r: $\frac{d}{dt}(p_r) = F_r = mr\dot{\theta}^2 - kr = m\ddot{r}$

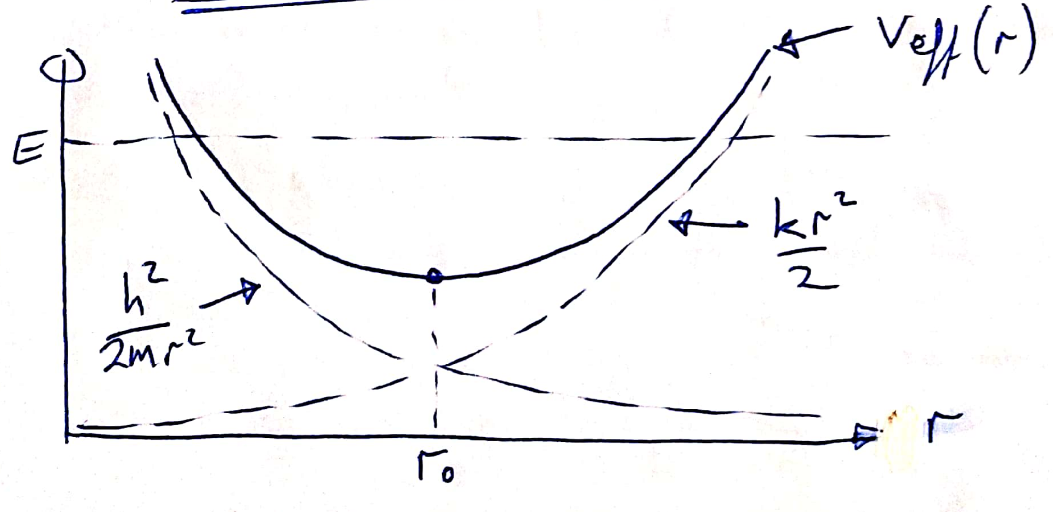
elim $\dot{\theta}$ using $mr^2\dot{\theta} = h$.

$$m\ddot{r} = \frac{mrh^2}{m^2r^4} - kr = \frac{h^2}{mr^3} - kr.$$

$$= -V'_{eff}(r)$$

thus. $V_{eff}(r) = \frac{h^2}{2mr^2} + \frac{kr^2}{2}$

c)



r_0 is where $V'_{eff}(r) = 0$ (or $m\ddot{r} = 0$)

$$\frac{h^2}{m r_0^3} = k r_0$$

$$\therefore r_0^4 = \frac{h^2}{mk}$$

and $\dot{\theta} = \frac{h}{m r_0^2} = \frac{\sqrt{mk}\, r_0^2}{m r_0^2} = \sqrt{\frac{k}{m}}$

d) $r = r_0 + \delta r \quad \therefore \ddot{r} = \ddot{\delta r}$

EoM: $m\ddot{r} = -V'_{eff}(r)$

$$m\ddot{\delta r} = -V'_{eff}(r_0 + \delta r) = -V'_{eff}(r_0) - V''_{eff}(r_0)\,\delta r + \ldots$$

(Taylor series)

but $V'_{eff}(r_0) = 0 \quad \therefore m\ddot{\delta r} = -V''_{eff}(r_0)\,\delta r$

$$\therefore \omega^2 = \frac{V''_{eff}(r_0)}{m}$$

but $V'_{eff}(r) = -\frac{h^2}{m r^3} + kr$ (part (b))

$V''_{eff}(r) = \frac{3h^2}{m r^4} + k$ where $h^2 = mkr^4$

so $V''_{eff}(r_0) = \frac{3mkr_0^4}{m r_0^4} + k = 4k$

$$\therefore \omega^2 = \frac{4k}{m}, \quad \omega = 2\sqrt{\frac{k}{m}}$$

i.e. oscillation at twice the orbital frequency ($\dot{\theta}$) giving elliptical orbits centred at the centre: