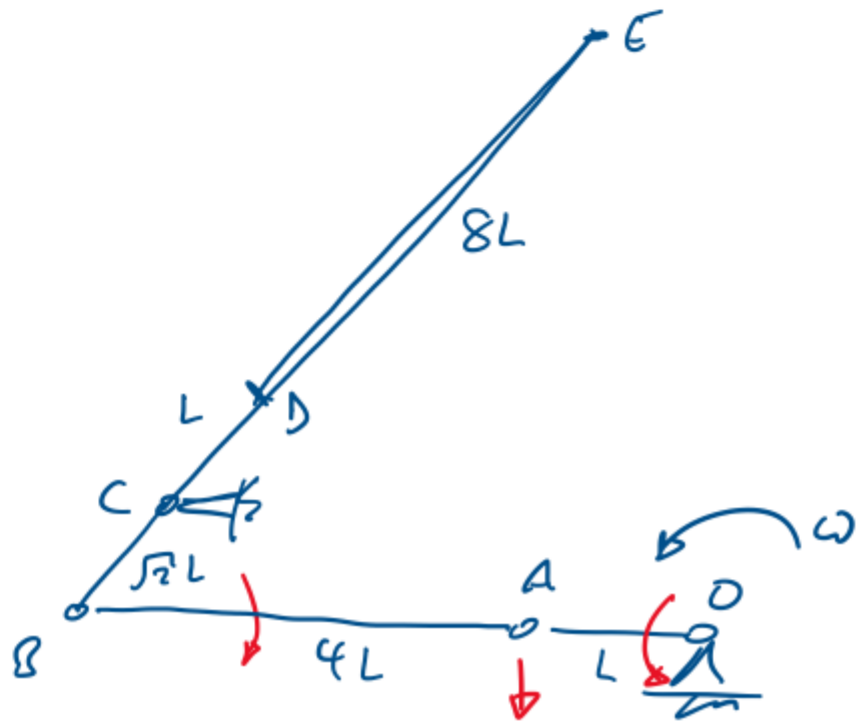
SOLUTIONS



(a) (i) IC_B @ B. (in line with OA & CB)

(ii) $v_A = \omega L$

$v_B = 0$

so $\omega_{AB} = \dfrac{\omega L}{4L} = \omega/4$

(iii) $P_{in} = P(v_G + v_D) + \sum_i Q_i \, \omega_{i,\text{relative}}$

$v_G = v_D = 0$ as $v_B = v_C = 0$.

Joints: O: $\omega_O = \omega$

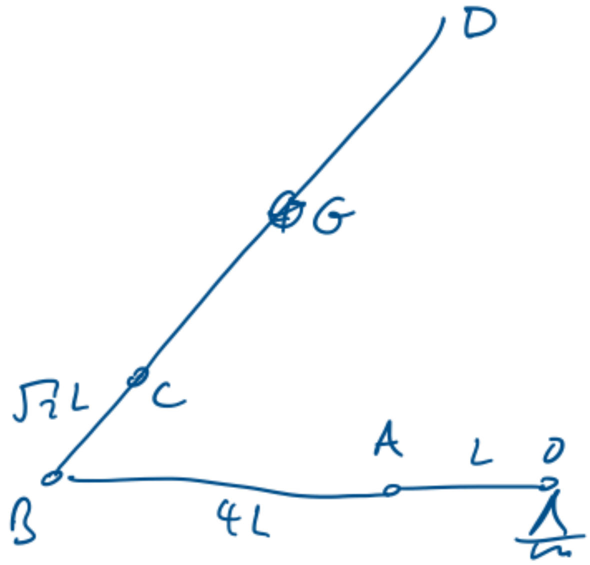
OA : anticlockwise
AB : clockwise.) $\Delta\omega_A = 5\omega/4$

BC : 0 rotation) $\Delta\omega_B = \omega/4$

C : 0 rotation

$$P_{in} = 0 + Q\omega\left(1 + \frac{5}{4} + \frac{1}{4}\right) = \frac{5}{2}Q\omega$$

(b) (i)

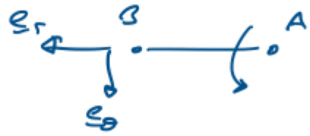


[Apply databook $\ddot{r}$ aligning $\hat{e}_r$ & $\hat{e}_\theta$ to each bar in turn]

$$\ddot{\underline{r}}_A = (\cancel{\ddot{r}} - r\underset{\omega}{\dot{\theta}^2})\,\underline{e}_r + (\cancel{r\ddot{\theta}} + \cancel{2\dot{r}\dot{\theta}})\,\underline{e}_\theta \quad \text{(databook)} \qquad \underline{e}_r \parallel OA$$

$$\ddot{\underline{r}}_A = r\omega^2\,\underline{i} = L\omega^2\,\underline{i} \quad \text{(ie circular motion).}$$

$$\ddot{\underline{r}}_B = \ddot{\underline{r}}_A + \ddot{\underline{r}}_{B/A}$$



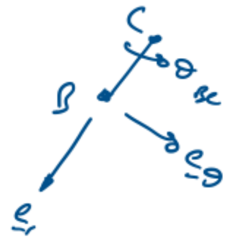
$$\ddot{\underline{r}}_{B/A} = \left(\cancelto{\ddot{r}=0}{\ddot{r}} - r\dot{\theta}_{AB}^2\right)\underline{e}_r + \left(r\ddot{\theta} + \cancelto{\dot{r}=0}{2\dot{r}\dot{\theta}_{AB}}\right)\underline{e}_\theta \qquad (e_r \parallel AB)$$

$$= -4L\left(\frac{\omega}{4}\right)^2 \underline{e}_r + r\ddot{\theta}_{AB}\,\underline{e}_\theta .$$

unknown.

$$\ddot{\underline{r}}_{B/C} = \left(\cancelto{\ddot{r}=0}{\ddot{r}} - \cancelto{\dot{\theta}_{BC}=0}{r\dot{\theta}_{BC}^2}\right)\underline{e}_r + \left(r\ddot{\theta} + \cancelto{\dot{r}=0}{2\dot{r}\dot{\theta}_{BC}}\right)\underline{e}_\theta \qquad (e_r \parallel CB)$$

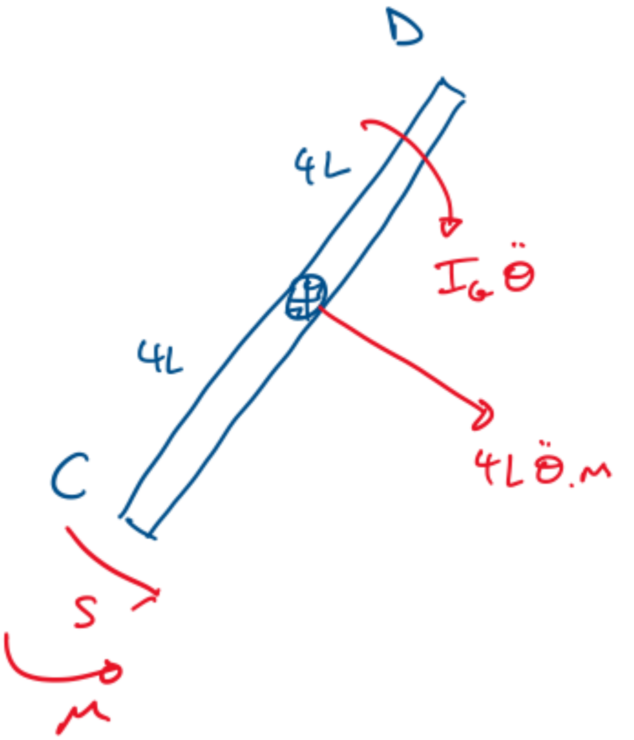
$= r\ddot{\theta}_{BC}\,\underline{e}_\theta$ ie in $(\underline{i} - \underline{j})$ direction.



So: $\ddot{\underline{r}}_B = L\omega^2 \underline{i} + \frac{L\omega^2}{4}\underline{i} - [\text{same}]\,\underline{j}$

$$= 5\omega^2 L/4\,(\underline{i} - \underline{j}) \;//$$

(iii) $\ddot{\theta}_{BC} = \dfrac{|\ddot{\underline{r}}_B|}{\sqrt{2}L} = \dfrac{5\cancel{L}\omega^2/4 \cdot \cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\cancel{L}} = \dfrac{5\omega^2}{4} \;/$



D'Alembert force & moment.

$$\sum M_C \Rightarrow \quad M = m\ddot{\theta}L^2 4^2 + \frac{m(8L)^2}{12}\ddot{\theta}$$

$$= mL^2\ddot{\theta}\left[16 + 16/3\right]$$

$$= \frac{64}{3}mL^2\ddot{\theta} = \frac{64}{3}\cdot\frac{5}{4}mL^2\omega^2 = \frac{80}{3}mL^2\omega^2$$

$$\simeq 26.7\,mL^2\omega^2$$

Q2

(a) (i) $\dot{\underline{e}}_1 = 0, \quad \dot{\underline{e}}_1^* = 0$ not rotating.

$$\dot{\underline{e}}_2 = \dot{\theta}\,\underline{e}_2^*$$

$$\dot{\underline{e}}_2^* = -\dot{\theta}\,\underline{e}_2$$

(ii) $$\underline{r}_A = \underline{r}_0 + r\,\underline{e}_2$$

$$\dot{\underline{r}}_A = \dot{\underline{r}}_0 + \dot{r}\,\underline{e}_2 + r\,\dot{\underline{e}}_2$$

$$= v\,\underline{e}_1 + r\dot{\theta}\,\underline{e}_2^*$$

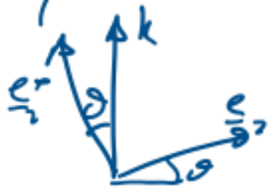
$$\ddot{\underline{r}}_A = a\,\underline{e}_1 + r\ddot{\theta}\,\underline{e}_2^* - r\dot{\theta}^2\,\underline{e}_2$$

(b) (i) For unit vectors ①: $\underline{\omega}_1 = \Omega\underline{k}$

②: $\underline{\omega}_2 = \Omega\underline{k} - \dot{\theta}\,\underline{e}_1$

$$\dot{\underline{e}}_1 = \underline{\omega}_1 \times \underline{e}_1 = \Omega \underline{k} \times \underline{e}_1 = \Omega \underline{e}_1^* \quad //$$

$$\dot{\underline{e}}_1^* = \underline{\omega}_1 \times \underline{e}_1^* = \Omega \underline{k} \times \underline{e}_1^* = -\Omega \underline{e}_1 \quad //$$



$$\dot{\underline{e}}_2 = \underline{\omega}_2 \times \underline{e}_2 = (\Omega \underline{k} - \dot{\theta} \underline{e}_1) \times \underline{e}_2$$

$$= \Omega \cos\theta \, \underline{e}_1 + \dot{\theta} \underline{e}_2^* \quad //$$

$$\dot{\underline{e}}_2^* = \underline{\omega}_2 \times \underline{e}_2^* = (\Omega \underline{k} - \dot{\theta} \underline{e}_1) \times \underline{e}_2^*$$

$$= -\Omega \sin\theta \, \underline{e}_1 - \dot{\theta} \underline{e}_2 \quad //$$

(ii) $\underline{r}_A = \underline{r}_0 + r \underline{e}_2$

$$\dot{\underline{r}}_A = r \dot{\underline{e}}_2 = r \Omega \cos\theta \, \underline{e}_1 + r \dot{\theta} \underline{e}_2^* \quad //$$

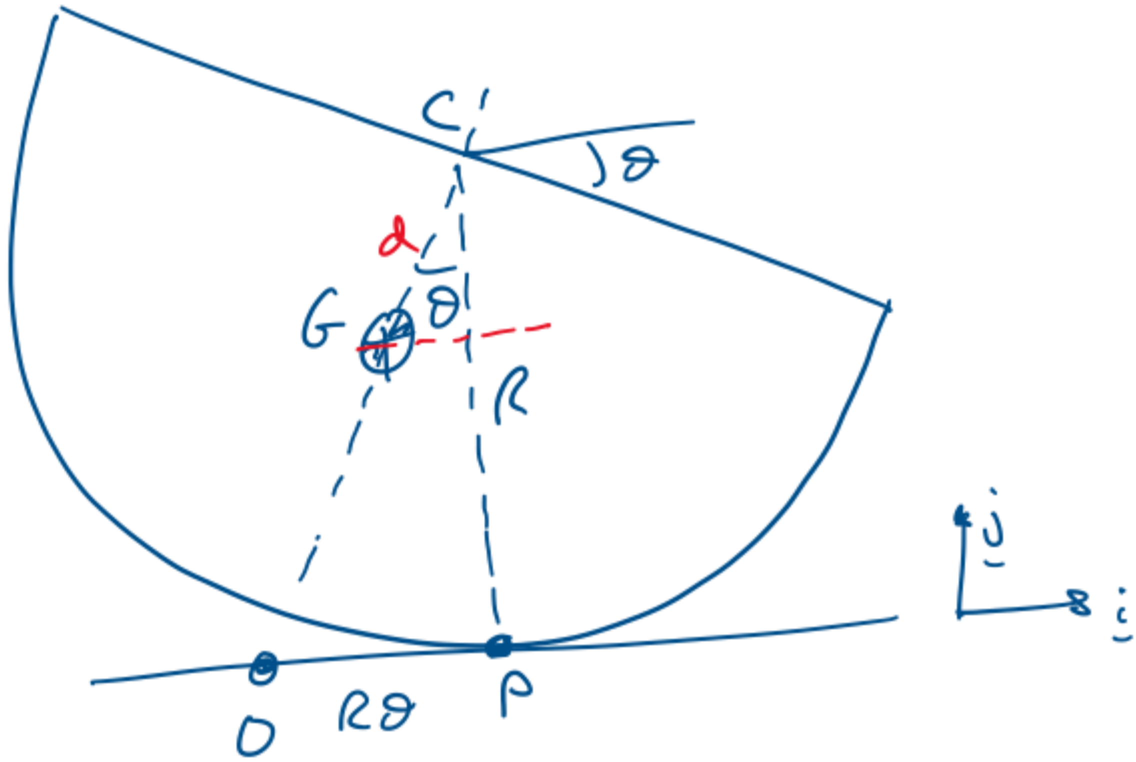
$$\ddot{\underline{r}}_A = r \dot{\Omega} \cos\theta \, \underline{e}_1 - r \Omega \dot{\theta} \sin\theta \, \underline{e}_1 + r \Omega \cos\theta \, \dot{\underline{e}}_1 + r \ddot{\theta} \underline{e}_2^* + r \dot{\theta} \dot{\underline{e}}_2^*$$

$$= r \dot{\Omega} \cos\theta \, \underline{e}_1 - r \Omega \dot{\theta} \sin\theta \, \underline{e}_1 + r \Omega \cos\theta . \Omega \underline{e}_1^* + r \ddot{\theta} \underline{e}_2^* + r \dot{\theta} (-\Omega \sin\theta \, \underline{e}_1 - \dot{\theta} \underline{e}_2)$$

$$= \underline{e}_1\left(r\dot{\Omega}\cos\theta - 2r\Omega\dot{\theta}\sin\theta\right) + r\Omega^2\cos\theta\,\underline{e}_1^* - r\dot{\theta}^2\underline{e}_2 + r\ddot{\theta}\,\underline{e}_2^*$$

(iii) Terms involving Ω: $\underline{e}_1$ & $\underline{e}_1^*$ components.
Opposite Haltere will cancel all these terms, so $m(r\Omega^2\cos\theta\,\underline{e}_1^*)$ is not felt.
The opposite '$\underline{e}_1$' terms will produce a torque that will most likely be dominated by the coriolis terms, torque $\simeq 4mr^2\Omega\dot{\theta}\sin\theta$ as Halteres are oscillating at high frequency so $\dot{\theta}$ is large.

Q3 (a)



$$\underline{r}_{G/C} = -d\sin\theta\,\underline{i} - d\cos\theta\,\underline{j}$$

$$\underline{r}_G = \underline{r}_C + \underline{r}_{G/C}$$

$$= R\theta\,\underline{i} + R\,\underline{j} + \underline{r}_{G/C}$$

$$= (R\theta - d\sin\theta)\,\underline{i} + (R - d\cos\theta)\,\underline{j}$$

$$= R\left(\theta - \tfrac{3}{8}\sin\theta\right)\underline{i} + R\left(1 - \tfrac{3}{8}\cos\theta\right)\underline{j}$$

$$\dot{\underline{r}}_G = R\left(\dot\theta - \tfrac{3}{8}\dot\theta\cos\theta\right)\underline{i} + \frac{3R}{8}\dot\theta\sin\theta\,\underline{j}$$

$$= R\dot{\theta}\left(1-\frac{3}{8}\cos\theta\right)\underline{i} + \frac{3R}{8}\dot{\theta}\sin\theta\,\underline{j}$$

(b) $V = \text{"}mgh\text{"} = mgd(1-\cos\theta)$

$$T = \frac{1}{2}m|v_G|^2 + \frac{1}{2}I_G\dot{\theta}^2$$

note: $|v_G|^2 = \dot{\theta}^2(R-d\cos\theta)^2 + \dot{\theta}^2 d^2\sin^2\theta$

$$= \dot{\theta}^2\left(R^2+d^2-2dR\cos\theta\right)$$

& $I_G = \frac{83R^2}{320}m$ (given)

so $T = \frac{1}{2}m\dot{\theta}^2\left(R^2+d^2-2dR\cos\theta\right) + \frac{1}{2}\cdot\frac{83R^2}{320}m\dot{\theta}^2$

$$= \frac{1}{2}mR^2\dot{\theta}^2\left(1+\left(\frac{3}{8}\right)^2 - 2\cdot\frac{3}{8}\cos\theta + \frac{83}{320}\right)$$

$$= mR^2\dot{\theta}^2\left(\frac{7}{10}-\frac{3}{8}\cos\theta\right)$$

(c) Small angles: $V \simeq mgd\cdot\frac{1}{2}\theta^2$

$$T \simeq MR^2\dot{\theta}^2\left(\frac{7}{10} - \frac{3}{8}\left(1 - \frac{1}{2}\theta^2\right)\right)$$

$$= MR^2\dot{\theta}^2\left(\frac{13}{40} + \frac{3}{16}\theta^2\right)$$

$$\simeq MR^2\dot{\theta}^2 \cdot \frac{13}{40} \quad // \quad \left(\text{as } \dot{\theta}^2\theta^2 \text{ is small}\right)$$

(d) $T + V = V_{max}$

$$MR^2\dot{\theta}^2\,\frac{13}{40} + Mgd\cdot\frac{1}{2}\theta^2 = V_{max}.$$

$$\dot{\theta}^2 = \frac{\frac{1}{2}\cancel{M}gd\left(\theta_0^2 - \theta^2\right)}{\cancel{M}R^2\cdot\frac{13}{40}}$$

$$\dot{\theta}^2 = \frac{20gd\left(\theta_0^2 - \theta^2\right)}{13R^2} \quad //$$

$$\cancel{2}\cancel{\dot{\theta}}\,\ddot{\theta} = \frac{20gd}{13R^2}\cdot\left(-\cancel{2}\theta\cancel{\dot{\theta}}\right)$$

$$\ddot{\theta} + \frac{20gd}{13R^2}\theta = 0$$

$$\ddot{\theta} + \frac{20}{13} \cdot \frac{3}{8} \cdot R \, \frac{g}{R^2} \, \theta = 0$$

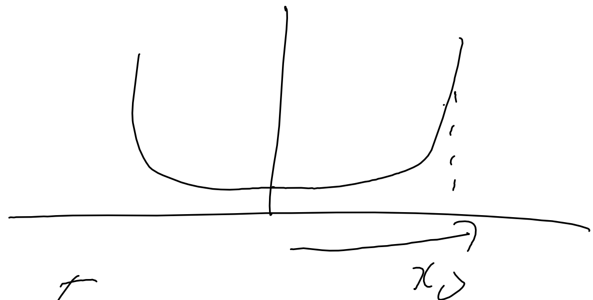
$$\ddot{\theta} + \frac{15}{26} \frac{g}{R} \theta = 0 \; //$$

(e) $$\omega_n^2 = \frac{15g}{26R}$$

$$\omega_n = \sqrt{\frac{15g}{26R}} \; //$$

①

$E = kx^4$



$$kx^4 + \frac{1}{2}m\dot{x}^2 = E$$

$$\frac{dx}{dt} = \dot{x} = \sqrt{(E - kx^4)\,2/m}$$

$$\int_0^{x_0} \frac{\sqrt{m}\,dx}{\sqrt{2}\sqrt{\underset{kx_0^4}{E} - kx^4}} = \int dt = \frac{T}{4}$$

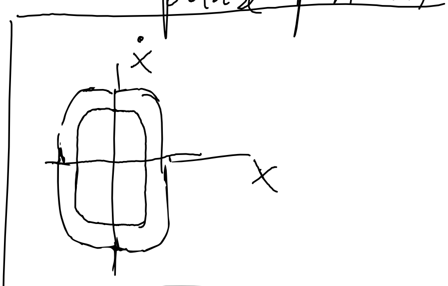
$$x = x_0 \tilde{x}$$
$$dx = x_0\, d\tilde{x}$$

$$\sqrt{\frac{m}{2}} \int_0^1 \frac{x_0\, d\tilde{x}}{\sqrt{kx_0^4 - kx_0^4 \tilde{x}^4}}$$

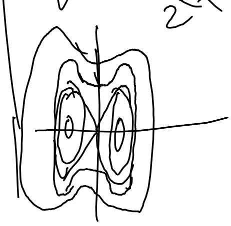
$$\sqrt{\frac{m}{2k}}\frac{1}{x_0}\underbrace{\int_0^1 \frac{d\tilde{x}}{\sqrt{1 - \tilde{x}^4}}}_{1.311}$$

period $T = 4\sqrt{\frac{m}{2k}}\frac{1}{x_0}$

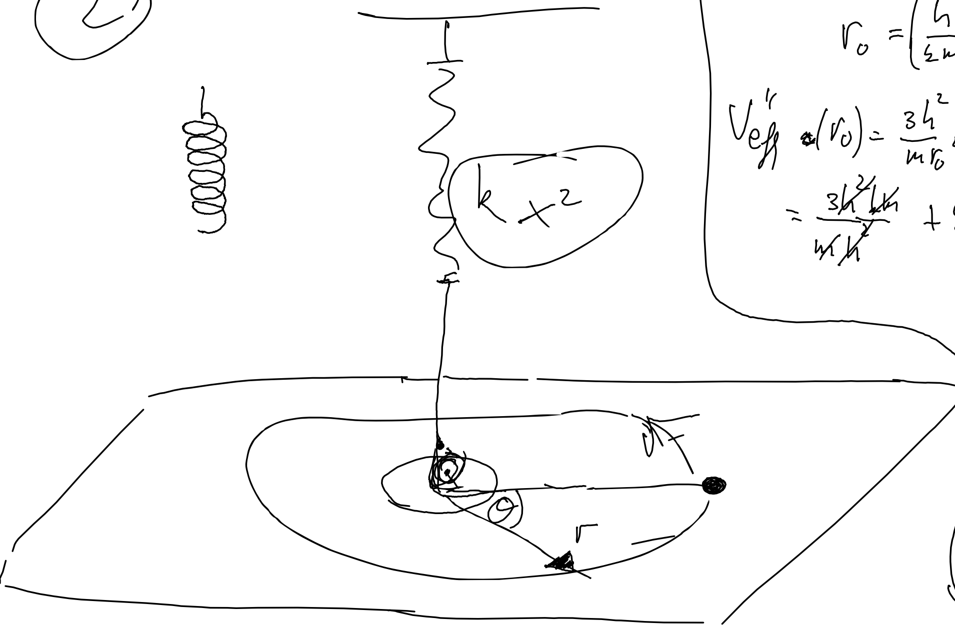
phase portrait



$V = -\frac{1}{2}kx^2 + kx^4$



(2)



$$V'_{eff} = 0 = -\frac{h^2}{m r_0^3} + k r_0$$

$$r_0 = \left(\frac{h^2}{km}\right)^{1/4}$$

$$V''_{eff}(r_0) = \frac{3h^2}{m r_0^4} + k$$

$$= \frac{3h^2 km}{m h^2} + k = 4k$$

$$\omega^2 = 4\frac{k}{m}$$

$$\omega = 2\sqrt{\frac{k}{m}}$$

$$\dot\theta = \frac{h}{m r_0^2}$$

$$= \frac{h}{m}\sqrt{\frac{km}{h^2}} = \sqrt{\frac{k}{m}}$$

$$T = \frac{1}{2} m\left(\dot r^2 + r^2 \dot\theta^2\right) \qquad V = \frac{1}{2} k r^2$$

orbits:

$$\frac{\partial L}{\partial \dot r} = m \dot r \qquad \frac{\partial L}{\partial \dot\theta} = m r^2 \dot\theta$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m r^2 \ddot\theta + m r \dot r \dot\theta = 0 \qquad \frac{\partial L}{\partial \theta} = \text{const} \Rightarrow \boxed{m r^2 \dot\theta = h}$$

$$\frac{\partial L}{\partial r} = m r \dot\theta^2 - kr \qquad \dot\theta = \frac{h}{m r^2}$$

$$m \ddot r = m r \dot\theta^2 - kr$$

$$= m r \frac{h^2}{m^2 r^4} - kr = \frac{h^2}{m r^3} - kr$$

$$V_{eff} = \frac{h^2}{2 m r^2} + \frac{1}{2} k r^2$$

3

$$L = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}l\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 - kx^2 + mgl\cos\theta$$

$$-mgl\sin\theta = m\ddot{x}l + ml^2\ddot{\theta}$$

$$-2kx = (m+M)\ddot{x} + ml\ddot{\theta}$$

$$\boxed{M = m}$$

$$\cancel{-\begin{pmatrix} 1 & l \\ 2m & ml \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & g \\ 2k & 0 \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix}}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + l\dot{\theta})^2$$

$$V = kx^2 - mgl\cos\theta$$

$$M\omega^2 A = KA$$

$$\det(K - \omega^2 M) = 0 \quad \begin{bmatrix} \omega^2 & \omega^2 l - g \\ \omega^2(m+M) - 2k & \omega^2 l \end{bmatrix}$$

$$\omega^2 = \frac{-2gm - 2kl \pm \sqrt{(2gm + 2kl)^2 - 8gklm}}{2lm}$$

large k limit

$$4g^2m^2 + 4k^2l^2 + \cancel{8gmkl - 8gklm}$$

$$a\left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}} \quad a \gg b$$

$$2kl\left(1 + \frac{1}{2}\frac{4g^2m^2}{k^2l^2}\right)$$

$$\omega^2 = \frac{g}{l} + \frac{k}{4} \frac{4g^2m^2}{k^2l^2}$$