Version FT/1

EGT1
ENGINEERING TRIPOS PART IB

Wednesday 5 June 2019   2 to 4.10

Paper 5

ELECTRICAL ENGINEERING

*Answer not more than **four** questions.*

*Answer not more than **two** questions from any one section and not more than **one** question from each of the other two sections.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

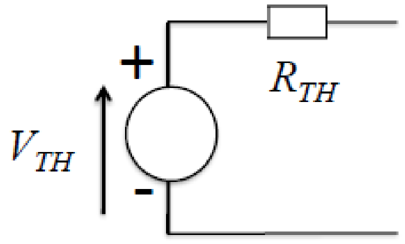
You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

Question 1

a)

Thévenin's theorem: *Any* <u>*linear*</u> *two-terminal network may be replaced by a voltage source* V_{TH} *in series with a resistance* R_{TH}.

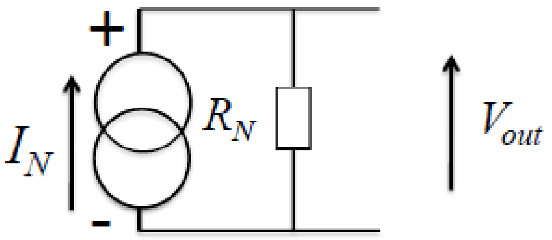
Thévenin's equivalent circuit



Here, V_{TH} is the open circuit voltage of the original circuit (V_{out}). R_{TH} = resistance measured between output terminals of original circuit.

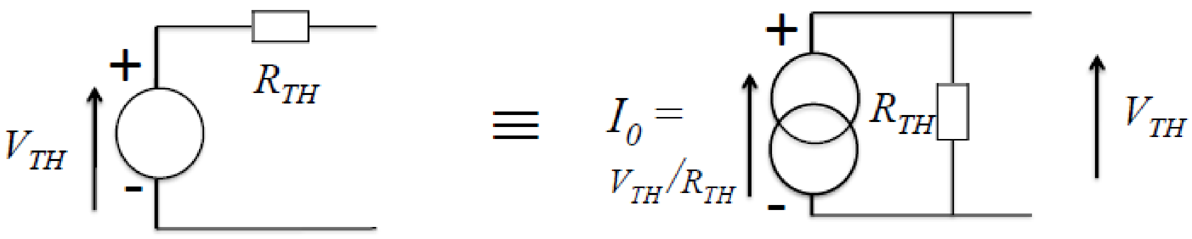
Norton's theorem: *Any* <u>*linear*</u> *two-terminal network may be replaced by a current source* I_N *in parallel with a resistance* R_N.

Norton's equivalent circuit



Here, I_N is the short circuit current of the original circuit. R_N = open circuit voltage/short circuit current.

Thevenin – Norton equivalent circuits



These equivalents can then be used to simplify more complex circuits and circuit elements.

b) i)

C_1 and C_2 are decoupling capacitors and correspond to an open circuit at DC operations.

The Transistor Q_1 is a NPN BJT and it operates as a current amplifier.

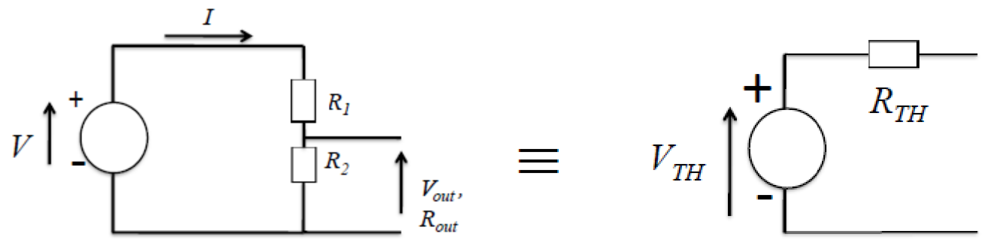
R_1 and R_2 constitute a potential divider, which is the polarization network to the base of the transistor.

R_4 increases the emitter voltage as a function of V_{be}, introducing a negative feedback on the output voltage.

R_3 sets the collector voltage and current.

ii)

Thevenin equivalent circuit can be used to rearrange the polarizing network composed by R_1 and R_2.



$V_{th}= V_{dd}\, R_2/(R_1+R_2)$; $R_{th}= R_1//R_2$

$I_B = V_{th}-V_B/R_{th}$; $V_E = V_B - 0.7$; $I_E = V_E/R_4 = (1+h_{FE})I_B$

$V_{CE}=V_C - V_E \Rightarrow V_C= V_{CE} + V_E= V_{CE} + V_B - 0.7$

$I_B= V_E/R_4\,(1+h_{FE})=V_{th}-V_B/R_{th}$ hence

$(V_{th} - V_E - 0.7)\, R_4\,(1+h_{FE}) = R_{th}\, V_E$

$(V_{th} - 0.7)\, R_4\,(1+h_{FE}) = V_E\,(R_{th} + R_4\,(1+h_{FE}))$

$V_E = (V_{th} - 0.7)\, R_4\,(1+h_{FE})/R_{th}+R_4\,(1+h_{FE})$

Then we also know that

$V_C = V_{cc} - I_C R_3=V_{CE}+V_E = V_{cc} - (h_{FE}\, V_E\, R_3)/(R_4\,(1+h_{FE}))$ therefore $V_{CE}= V_{cc} - V_E\,(1+(h_{FE}\, R_3)/(R_4\,(1+h_{FE}))$

Hence,

$V_{CE} = V_{cc} - (R_4\,(1+h_{FE}) + h_{FE}\, R_3)\,(V_{th} - 0.7)/(R_{th} + R_4\,(1+h_{FE}))$

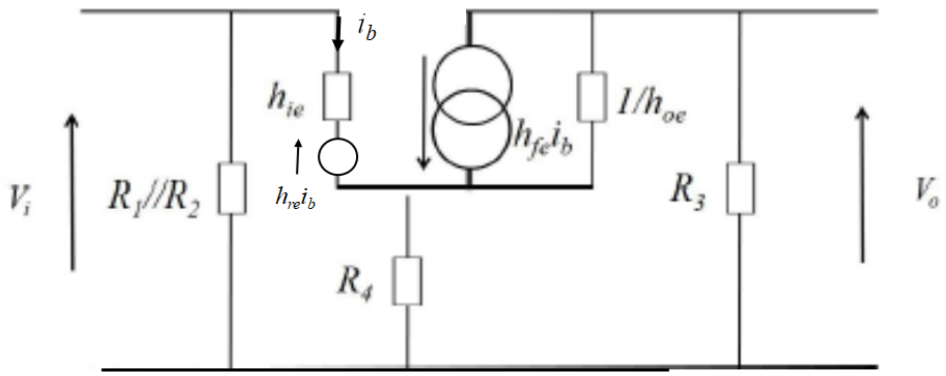
We know that:

$V_{th} = 10\, R_2/(R_1+R_2) = 1.75V$; $R_{th}=R_1//R_2 = 8.12\ k\Omega$

$h_{FE} = 50 \Rightarrow V_{CE} = 4.56V$

$h_{FE} = 300 \Rightarrow V_{CE} = 3.86V$ variation is ~15.4%

c)



d)

Ignore hre and 1/hoe

$V_i = V' + h_{ie}\, i_b$

$V' = R_4\,(1+h_{fe})\, i_b$

$V_0 = -h_{fe}\, i_b\, R_3$; $V_i = i_b\,(h_{ie} + R_4\,(1+h_{fe}))$

$V_0/V_i = (-h_{fe}\, R_3)/(h_{ie} + R_4\,(1+h_{fe}))$

e)

R_4 sets the emitter voltage and it is required to create a negative feedback effect hence stabilising the circuit bias.

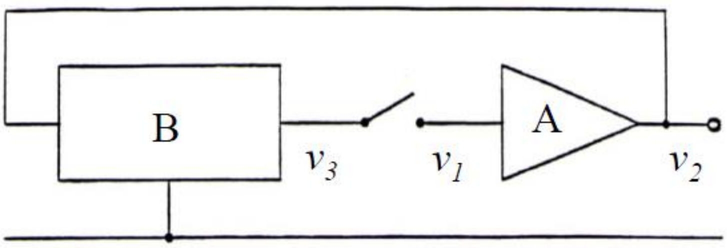
The presence of R_4 also reduces the overall gain of the amplifier.

A bypass capacitor in parallel to R_4 can be used to mitigate its effect at the operation frequency while preserving the benefits in DC.

Very popular question attempted by almost everyone. a) Done well although most answers referred to resistance rather than impedance. About half forgot to answer the last part about how they can be used to simplify circuits. b) i) Quite a variety of different explanations, but most got the roles of the resistors pretty much right with a few odd answers. Many answers lost marks for not explaining what C1 and C2 were in the circuit for. b) ii) The hardest section of the question. There were a wide variety of approaches, which is unusual as it is basically taken directly from the 1B examples papers. The majority got the use of the Thevenin equivalent for the base biasing but not many managed to pull together the equations for the variation in VCE. A few erroneously assumed that IB was negligible. c) Well answered, but many were caught by the fact that they should have included hre and hoe as they were not neglected. Several missed vin vout and ib. d) Very well answered, almost all got this right. e) Also well answered

Question 2

In order to generate oscillation positive feedback is required.



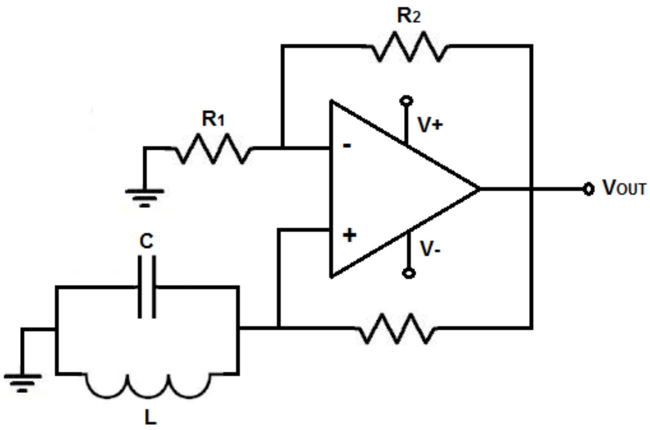
If AB = -1, then the gain becomes infinite, and oscillations will occur.
The circuit can produce an output without any external input.
This can be exploited to make oscillators using circuits which give AB = -1 at a particular frequency

Suppose an external sinusoidal input is applied to the amplifier input.
If the voltage at the output of the passive network is the same as this voltage, the switch can be closed and an output will be maintained with the original input removed.
Conditions for oscillation are:

$$\angle AB(j\omega) = 0 \text{ or } 2\pi \qquad |AB(j\omega)| = 1$$

Loop phase of 0 or 2π — Loop gain of unity

If we have an ideal Op-Amp: $V^+ - V^- = 0V$; and $I^+ = I^- = 0A$.



So, we have at the "–" node of the Op-Amp

$I_1 = -V^+/R_1 = V^+ - V_o/R_2$ => $V+ = R2/(R1+R2)\ V_o$

At the "+" node of the Op-Amp

$-V+/(L//C) = (V+ - V_o)/R$ where $L//C = j\omega L/(1-\omega^2 LC)$

$-V+R = j\omega L\ (V+ - V_o)/(1-\omega^2 LC)$

Rearranging we get:

$V_o\ (j\omega L/(1-\omega^2 LC)) = V+\ \ R(1-\omega^2 LC) + j\omega R/(R(1- \omega^2 LC))$

So isolating V^+ we have:

$V^+ = V_o\ j\omega LR/(R(1-\omega^2 LC)+j\omega LC) = V_o\ R/(R+ R/j\omega L(1-\omega^2 LC))$

Imposing phase zero means $1=\omega^2 LC$, $\omega=1/\sqrt{LC}$ which means that $V^+ = V_o$.

So consequently we have $R_2/(R_1+R_2)=1$.

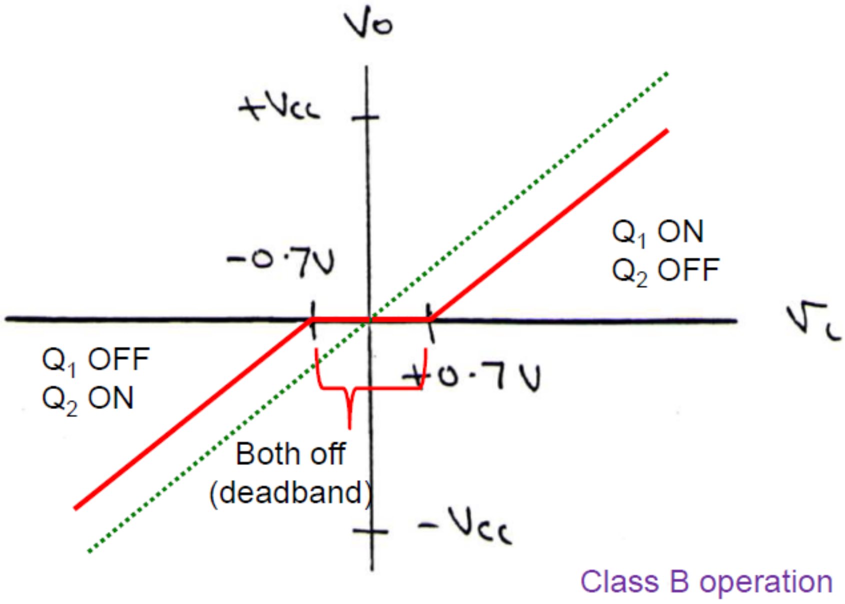
If $R_2/(R_1+R_2)<1$ then we have decreasing oscillation towards attenuation.

$R_2/(R_1+R_2)>1$ we have an over-increase of oscillation.

c)

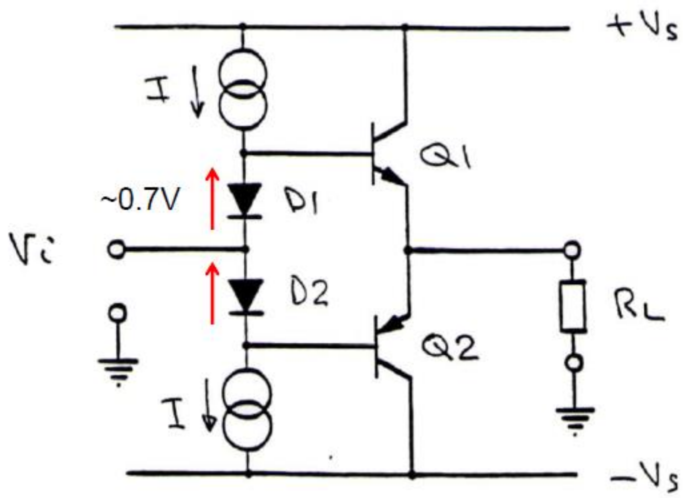
The circuit is a class B amplifier using BJTs. This configuration provides a larger output swing and minimizes the power dissipation (active).

A drawback is the fact that the activation voltage of the BJTs (~ 0.7V) generates a crossover distorsion in the output signal as shown below.



This creates extra harmonics in the signal at the output causing distortions.

A solution to this is the Class AB amplifier, which required the use of two diodes across the bases of Q1 and Q2. Bias for the diodes is provided by two current sources.



Much less popular question probably as it was on oscillators which is a less known section of the course. Answers overall were good. a) Done well as a rule, with some getting positive and negative feedback confused. A few used a feedback calculation to explain stability. b) Well answered with most getting the derivation of the complex gain right and then getting the resonant frequency. A few less clear answers for the level control section with many not realising the role of the non-inverting gain section at resonance. c) Well answered overall as this was pretty much book work describing the problems of crossover distortion and its effects on the frequency components of the oscillator.

SECTION B

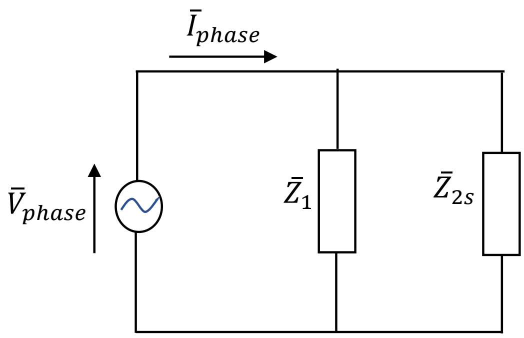
*Answer not more than **two** questions from this section*

A three-phase 415 V, 50 Hz supply is connected to a balanced star-connected load and a balanced delta-connected load. The impedance of each leg of the delta-connected load is 300+j150 Ω and each phase of the star-connected load consists of a 400 Ω resistor in parallel with a 10 μF capacitor.

(a) Can we analyse this system by an equivalent one-phase circuit? If yes, draw the most simplified structure of the corresponding one-phase circuit. If no, explain the reason. [4]

Solution:

Yes. Since both loads are balanced they will consume same P and Q in each phase. Thus, an analysis of one-phase circuit will suffice. The system can be analysed by an equivalent one-phase circuit as follows.



For the star-connected supply: $V_{phase} = \frac{415}{\sqrt{(3)}} = 240$ V.

For the star-connected load: $\bar{Z}_1 = R||\frac{1}{j\omega C} = \frac{R}{1+j\omega C} = \frac{400}{1+j2\pi\times 50\times 10\times 10^{-6}\times 400}$
$= 155.19 - j194.91\ \Omega$

For the delta-connected load: $\bar{Z}_2 = 300 + j150\ \Omega$

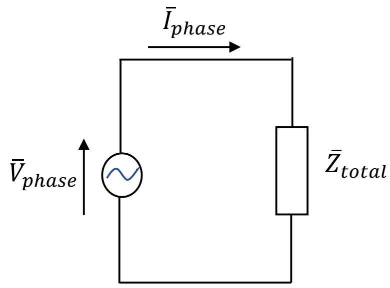
Then, $\bar{Z}_{2s}$ is derived by transforming delta load to star as follows.

$$\bar{Z}_{2s} = \frac{\bar{Z}_2}{3} = \frac{300 + j150\ \Omega}{3} = 100 + j50\ \Omega$$

The one-phase circuit can be simplified to

where $\bar{Z}_{total} = \frac{\bar{Z}_1\bar{Z}_{2s}}{\bar{Z}_1+\bar{Z}_{2s}} = 94.6 + j7.75\ \Omega$

(b) Calculate



(i) the line current; [4]

Solution:

$$\bar{I}_{phase} = \frac{\bar{V}_{phase}}{\bar{Z}1} + \frac{\bar{V}_{phase}}{\bar{Z}_{2s}} = 2.52 - j0.21 \text{ A} = 2.53\angle - 4.68^\circ \text{ A}$$

In this circuit, the phase and line current are equal.

(ii) the input power; [4]

Solution:

$$p = 3V_{phase}I_{phase}Cos\phi = 3 \times 240 \times 2.53 \times \cos(-4.68) = 1815.53 \text{ W}$$

(iii) the input VARs. [4]

Solution:

$$Q_{load} = 3I^2_{phase}X = 3 \times 2.53^2 \times 7.75 = 148.82 \text{ VAR}$$

(c) We wish to alter the power factor of the system to 0.95 lagging.

(i) Explain whether delta-connected capacitors or inductors must be used. [2]

Solution:

Note that capacitors and inductors do not affect P. Thus, the required Q to reach power factor (PF) equal to 0.95 lagging can be calculated as follows.

$$Q_{new} = \frac{P}{PF}\sin(\cos^{-1}(PF)) = \frac{1815.53}{0.95}\sin(\cos^{-1}(0.95)) = 596.75 \text{ VAR} \quad (1)$$

The total reactive power must change from 148.82 VAR to 596.75 VAR. Thus, inductors are required.

(ii) Derive the required value of the capacitors/inductors. [5]

Solution:

For star-connected inductors we have $X_s = \frac{V^2_{phase}}{Q_s} = \frac{240}{596.75-148.82} = 128.59\ \Omega$. Thus, for delta-connected inductors we need $X_D = 3X_s = 385.77\ \Omega = 2\pi 50 H_D$. Thus, we the value of delta-connected inductors must be $H_D = 1.23$ H.

(iii) Explain whether each of the following parameters change after correcting the power factor. [2]

A. The total dissipated power

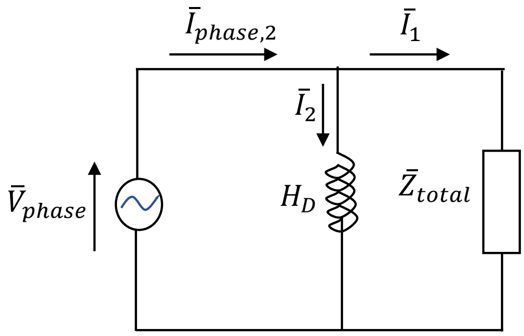
Solution:

The total dissipated power is not affected as capacitors only generate reactive power.

B. The supply current

Solution:

The supply current changes as follows.



$I_{phase,2} = I_1 + I_2$, where $I_1 = I_{phase}$, i.e., the phase current before correcting the power factor.

This was a straightforward question in three phase systems and altering the power factor. Students has shown a great understanding of the one phase equivalent of the system. The most common error in first part of this question was on dealing with complex numbers as some of students have only derived the absolute value of currents and added them without taking care of their magnitude. While deriving the input power and VARs in part b, the majority of students were successful, however, a small group of them were confused with line/ phase current and voltage. Part c of the question was a bit challenging as opposed to what the students expect, i.e., correcting PF with using capacitors, the question was focused on impact of adding inductors. The overall performance of students was good in this question as the average mark indicates.

(a) Explain why 3 phase power generation is superior to other number of phases. [4]

Solution:

(a) With single phase, even though less wire is used, efficiency is very low. Two phase requires the same number of wires as three phase due to three phase not needing a neutral wire, but it has lower efficiency. Higher number of phases increase efficiency only marginally but extra wires are required. Hence three-phase is adopted throughout the world.

(b) A synchronous generator is connected to an infinite bus.

(i) What is an infinite bus? Give an example where infinite bus assumption may not hold. [3]

(ii) What conditions must be satisfied for a synchronous AC generator to produce steady torque? Explain why these conditions must be met. [4]

Solution:

(i) Infinite bus means fixed frequency and fix voltage. The bus is so strong that no single generator can change neither frequency nor voltage. If generators are not connected to national grid, infinite bus assumption may not hold.

(ii) Condition 1: The rotor and stator fields must have the same number of poles. Condition 2: Rotor speed must be equal to the speed of the stator driven field. If these conditions are not met, torque will oscillate sinusoidally with zero average.

(c) A synchronous star-connected generator with 4 poles and a synchronous reactance of $X_s = 0.2\Omega$, is connected to a 50 Hz infinite bus with line voltage 11 kV.

(i) Find the speed of rotation and the torque of the prime mover if the prime mover power is set to 400 MW. [2]

(ii) If the prime mover is set to 400 MW and the power factor at the terminals of the machine is 0.9 lagging, find the excitation voltage. [6]

(iii) If the prime mover power reduces to 250 MW with line voltage and the excitation voltage that you obtained in (ii) remaining unchanged, calculate the new power factor. [6]

Solution:

(i)

$$\omega_s = \frac{\omega}{p} = \frac{2\pi f}{p} = \frac{100\pi}{2} = 157.1 \text{ rad/s}$$

 (cont.

$$T = \frac{P}{\omega_s} = \frac{400 \times 10^6}{157.1} = 2550 \text{ kNm}$$

(ii)

$$P = 3I_{ph}V_{ph}\cos\theta \Longrightarrow I_{ph} = \frac{P}{3\frac{V_l}{\sqrt{3}}\cos\theta} = 23.3 \text{ kA}$$

$$E = \sqrt{(V_{ph} + I_{ph}X_s\sin\theta)^2 + (I_{ph}X_s\cos\theta)^2} = 9.37 \text{ kV} = 16.2 \text{ kV}_{\text{line}}$$

(iii)

$$I_{ph}\cos\theta = \frac{P}{3V_{ph}} = \frac{250 \times 10^6}{3\frac{11\times10^3}{\sqrt{3}}} = 13.1 \text{ kA}$$

$$I_{ph}\sin\theta = \frac{\sqrt{E^2 - (I_{ph}X_s\cos\theta)^2} - V_{ph}}{X_s} = 13.2 \text{ kA}$$

$$I_{ph} = \sqrt{I_{ph}^2\cos^2\theta + I_{ph}^2\sin^2\theta} = 18.6 \text{ kA}$$

$$\cos\theta = \frac{13.1}{18.6} = 0.704$$

This question was the least popular question, probably because it spanned a variety of topics. It required information on basics of transmission and synchronous motors (part a and b) as well as some calculations regarding the synchronous motors (part c). The answers to the basic information questions were available in the lecture notes. The students were overall successful. The great majority of them showed that they grasped the basics on both topics. They were also quite successful with part c. The common error in part c was wrong use of geometry. I believe this is due to the time limitations, so if they had more time, the great majority of the students could have achieved near perfect marks.

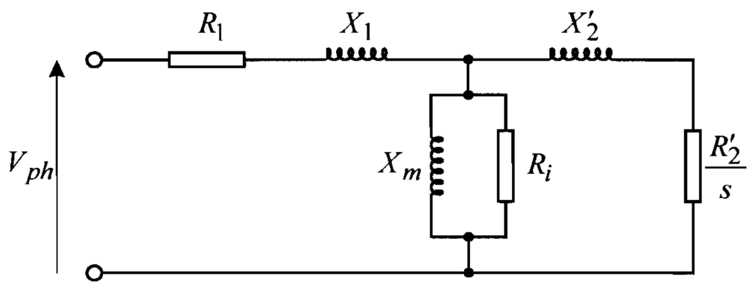
(a) Explain why the induction motor is sometimes known as an asynchronous motor. [3]

Solution: If rotor moves at the synchronous speed, it will see a stationary stator field. Therefore emf induction will not occur and no torque is produced. At any speed other than synchronous speed torque is produced.

(b) Consider an induction motor.

(i) Draw its equivalent circuit model. [2]

Solution:



R_1: Stator winding resistance

X_1: Stator winding reactance

R_2: Rotor winding resistance

X_2: Rotor winding reactance

X_M: Magnetising reactance

R_i: Iron core resistance

(ii) What is the locked rotor test? How is the equivalent circuit approximated? [3]

Solution:

Locked motor test prevents the rotor from rotating, hence slip approaches 1. As a result parallel components can be ignored.

(iii) What is the no load test? How is the equivalent circuit approximated? [3]

Solution:

Rotor rotates freely with no load. No torque is generated, hence slip approaches 0. As a result, series components can be ignored.

(c) A four-pole star-connected, three-phase induction motor is operated at 1350 rpm when driven by a line voltage of 600 V at 50 Hz. The equivalent circuit of this induction motor has the following circuit parameters: $R_1 = R'_2 = 2\ \Omega$, $R_0 = 2500\ \Omega$, $X_m = 500\ \Omega$, $X_1 = X'_2 = 6\Omega$.

 (cont.

(i) Calculate the value of slip [3]

Solution:

$$N_s = \frac{60 \times 50}{2} = 1500 \text{ rpm, thus, } \omega_s = 157.1 \text{ rad/s}$$

$$\text{slip} = \frac{N_s - N_r}{N_s} = \frac{1500 - 1350}{1500} = 0.1$$

(ii) Stating any approximations, calculate the peak torque. [5]

Solution:

Since $\frac{2}{s} + j6 << R_0||jX_m$, ignore R_0 and X_m.

$$V_{ph} = \frac{600 \text{ V}}{\sqrt{3}} = 346.41 \text{ V}$$

$$S_m = \frac{R_2'}{Z_s}$$

$$Z_s = \sqrt{R_1^2 + (X_1 + X_2')^2} = 12.16\ \Omega$$

$$I_2' = \frac{V_{ph}}{\sqrt{(R_2'/s + R_1) + X_1 + X_2'}}$$

$$I_{2m}' = \sqrt{\frac{V_{ph}^2}{2Z_s(Z_s + R_1)}} = 19.63 \text{ A}$$

$$T_{max} = \frac{3}{\omega_s}\frac{I_{2m}'^2 R_2'}{S_m} = \frac{3V_{ph}^2}{2\omega_s(Z_s + R_1)} = \frac{3 \times 346.41^2}{2 \times 157.1 \times (12.16 + 2)} = 80.91 \text{ Nm} \quad (2)$$

(iii) Find the speed at which the maximum torque occurs. [3]

Solution:

$$S_m = \frac{R_2'}{Z_s} = 0.16$$

The speed at which peak torque occurs is $(1 - S_m) \times 1500 = 1260$ rpm

(iv) How can we maximize the starting torque of this motor? Derive the required changes in the circuit elements of the motor. [3]

Solution: By adding extra resistance to the motor, we can maximize the starting torque, i.e., set the speed at which peak torque occurs to 0.

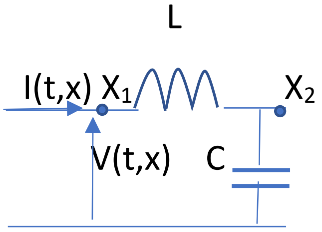
$$1 - S_m = 0 \longrightarrow S_m = 1 \longrightarrow \frac{R_{2,new}'}{Z_s} = 1 \quad (3)$$

$$R_{2,new}' = Z_s \longrightarrow Z_s = 12.16\ \Omega = 2 + R_{2,extra}' \quad (4)$$

$$R_{2,extra}' = 10.16\ \Omega$$

This question was a popular question as well. It focused on some basic theory on induction motors (part a and b), and some basic calculations on the induction motors (part c). All students had some understanding on the basics of induction motors and almost all of them got near perfect scores on part b, which was equivalent circuits and locked-motor and no-load tests. In part c, students needed to have an engineer's mindset and ignore R_0 and X_m as they are much larger than other components. The ones that have missed this point spent lots of time for calculations and could not derive the correct answer due to calculation errors

Answer Q6



(a) $\delta V = -L\,(X_2 - X_1)\frac{\partial I}{\partial t} = -L\delta x\frac{\partial I(x,t)}{\partial t}$, $\partial V/_{dx} = -L\,\partial I/_{\partial t}$ (1)

$\delta I = -C\,(X_2 - X_1)\frac{\partial V(x,t)}{\partial t} = -C\delta x\frac{\partial V(x,t)}{\partial t}$, $\partial I/_{dx} = -C\,\partial V/_{\partial t}$ (2)

Taking the 2nd differential of V and I in x

$$\frac{\partial^2 V}{\partial x^2} = -L\,{}^{\partial(\frac{\partial I}{\partial x})}/_{\partial t}$$

Substituting for *dI/dx*

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$$

Similarly takingthe 2nd differential of I in X

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}$$

Both V and I are functions of distance and time i.e V= V(*x*,*t*) and I= I(*x*,*t*). As the equations take the form of the 2nd differential space being proportional to 2nd differential in time, they describe wave equations. Hence V and I behave like waves over the two parallel wires.

(b) General form of the solution to the wave equation is:

$$V(x,t) = V_1 e^{j(\omega t - \beta x)} + V_2 e^{j(\omega t + \beta x)}$$

The solution allows for waves which travel in the forward x or -x directions (we do not do the same in time as we consider negative going time to be unphysical). V_1 and V_2 can in turn have magnitude and phase, hence they are represented in phasor form. The actual measurable voltage is the real component of the overall complex voltage.

$$V(x,t) = Re\,(\hat{V}_1 e^{j(\omega t - \beta x)} + \hat{V}_2 e^{j(\omega t + \beta x)})$$

Similarly

$$I(x,t) = Re\,(\hat{I}_1 e^{j(\omega t - \beta x)} - \hat{I}_2 e^{j(\omega t + \beta x)})$$

But in the negative x direction the current is taken to be reversed

(c) Using the general solution for voltage in the wave equation one gets

$$-\beta^2 = -\omega^2 LC, \quad \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

From (1) above, considering only the forward going wave, one gets

$$-j\beta V(x,t) = -jL\omega I(x,t), \quad \frac{V(x,t)}{I(x,t)} = L\frac{\omega}{\beta}$$

Substituting for the forward going wave from the equation above

$$\frac{\hat{V}_1(x,t)}{\hat{I}_1(x,t)} = \sqrt{\frac{L}{C}} = Z_0$$

We define the ratio of voltage to current as an impedance having units ohms. It is not a physical impedance as it is not taken across the same two points, but an apparent impedance for the two parallel wires seen by the voltage source, termed the characteristic impedance. Note that it is a real number and constant in both *x* and *t* for defined *L* and *C* of the two wire system.

For the backward going wave

$$\frac{\hat{V}_2(x,t)}{\hat{I}_2(x,t)} = -\sqrt{\frac{L}{C}} = -Z_0$$

(d) Since $\frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = 1.0X10^8 \;\; \omega = 2\pi 10^7 \;\; \beta = 2\pi/\lambda \;\; \lambda = 10$ m

Considering the 1% variation,

$$V(x,t) = 0.99V(0,t), Re(\, e^{-j\beta x}) = 0.99, \;\; cos\beta x = 0.99, \beta x = \cos^{-1} 0.99 \;\; and\; x = 0.23\, m$$

-

(e) (i) The reflection co-efficient at the load is:

$\bar{\rho} = \frac{\bar{Z}_L - Z_0}{\bar{Z}_L + Z_0}, \; Z_0 = \sqrt{\frac{4.0x10^{-7}}{2.5x10^{-10}}} = 40\Omega \;\; \bar{z}_L = 50 + J10\Omega$

$\bar{\rho} = \frac{10+j10}{90+J10} = 0.16 < 38.7$ deg

(e) (ii) The source voltage is ideal , i.e Z_S=0, and has phasor value $\widehat{V_s} = 5 < 0$

Along the transmission line the source voltage varies with *x* according to

$$\widehat{V_F}(x) = \widehat{V_F} e^{-j\beta x}$$

Where $\widehat{V_F}$ is the forward going voltage.

At the load situated at x=L there is a reflected voltage

$$\widehat{V_R}(x) = \bar{\rho}\widehat{V_F} e^{-j\beta L} e^{j\beta(x-L)} = \bar{\rho}\widehat{V_F} e^{j\beta(x-2L)}$$

X=0 is taken to be at the source voltage connection to the transmission line. There is therefore a phase shift which must be considered between the forward and backward going wave which gives rise to the – 2βL term in the exponent.

Or one can term $\hat{V}_B = \bar{\rho}\widehat{V_F} e^{-j\beta 2L}$ $and \; V(x) = \hat{V}_F e^{-j\beta x} + \hat{V}_B e^{j\beta x}$

When the reflected voltage reaches the source end i.e. x=0

$$\widehat{V_R}(0) = \bar{\rho}\widehat{V_F} e^{-j\beta 2L}$$

The voltage x=0 is $\hat{V}_{in} = \widehat{V_s}$ as Z_S=0, hence

$$\hat{V_s} = \widehat{V_F}(0) + \widehat{V_R}(0) = \widehat{V_F}(1 + \bar{\rho}e^{-j\beta 2L})$$

$$\hat{V_F} = \frac{\hat{V_s}}{(1 + \bar{\rho}e^{-j\beta 2L})}$$

The general voltage along the line is given as: $\hat{V}\,(x) = \widehat{V_F}(x) + \widehat{V_R}(x)$

Therefore $\hat{V}\,(x) = \frac{\hat{V_s}}{(1+\bar{\rho}e^{-j\beta 2L})}\,(e^{-j\beta x} + \bar{\rho}e^{j\beta(x-2L})$

$\hat{V}(x) = Re(\,\hat{V}(x)\,)$(which oscillates in ωt hence the phasor notation).

(e) (iii)

The reflection co-efficient at the load is 0.16<38.7

The voltage at the load *x = L* is therefore

$$Re\left(\hat{V}(L)\right) = Re\left(\,\hat{V}_F e^{-j\beta L} + \hat{V}_F\bar{\rho}e^{-j\beta L}\right) = Re(\,\hat{V}_F e^{-j\beta L}(1 + \bar{\rho}))$$

$\beta = 2\pi/10\ \ m^{-1}$ and L= 5m $\beta L = \pi$ and $2\beta L = 2\pi$

$$Re\left(\hat{V}(L)\right) = Re\left(\frac{5}{(1+\bar{\rho}e^{-j2\pi})}.\,e^{-j\pi}(1 + \bar{\rho})\right) = -5\text{ V}$$

$$\hat{V}(L) = -\hat{V_s}$$

The load voltage is 180 deg out of phase with the source voltage, the time domain signal is:

$$V(t, L) = 5\cos(\omega t - \pi)\,V$$

A question on transmission lines. The first part (a- c) examined the understanding of the fundamental theoretical concepts and the implication of having spatially distributed voltage and current signals. Most of the candidates attempting the question were able elucidate the existence of voltage and current waves in transmission lines and the consequence of having reflected waves. Parts d and e we problem based around microstrip connections on a printed circuit board. Part d was a practical and simple estimation of the length of the microstrip connection above which transmission line behaviour should be considered. Surprisingly many students were unable to do this, a direct consequence of which is the low average mark. Those who did understand what was required, about half those attempting, fell over in not being able to solve a very simple complex exponential equation. There seems to be an almost universal unfamiliarity with De Moirve's expansion of a complex exponential and considering only the real part. Many tried to solve it by taking natural logarithms. Part d was a substantial problem (in 3 parts) on transmission line propagation. Most could calculate the reflection co-efficient at the load. The better students (10%) made a good attempt at deriving the standing wave expression and the voltage at the load.

Question 7

a) $\overline{\nabla} X\overline{E} = -\frac{\partial}{\partial t}\overline{B}$ and $\overline{\nabla} X\overline{H} = \overline{J} + \epsilon_o \frac{\partial}{\partial t}\overline{D}$

Also $\overline{B} = \mu_o \overline{H}$

$$\overline{\nabla} X(\overline{\nabla} X\overline{E}) = -\frac{\partial}{\partial t}(\overline{\nabla} X\overline{B})$$

Using the vector identity

$$\overline{\nabla} X(\overline{\nabla} X\overline{E}) = \overline{\nabla\nabla}.\overline{E} - \overline{\nabla^2}\overline{E}$$

And noting that $\overline{\nabla}.\overline{E} = \epsilon_o(\overline{\nabla}.\overline{D}) = \rho$ and that $\rho = 0$ and J = 0 in free space

$$\overline{\nabla^2}\overline{E} = \mu_o \frac{\partial}{\partial t}(\overline{\nabla} X\overline{H})$$

Substituting the 2nd Maxwell equation

$$\overline{\nabla^2}\overline{E} = \epsilon_o\, \mu_o \frac{\partial^2}{\partial t^2}\overline{E}$$

Similarly noting that $\overline{\nabla}.\overline{B} = \mu_o(\overline{\nabla}.\overline{H}) = 0$

$$\overline{\nabla^2}\overline{H} = \epsilon_o\, \mu_o \frac{\partial^2}{\partial t^2}\overline{H}$$

These are both wave equations. Hence the electric field and magnetic field propagate through free space as waves when there is a time varying electric or magnetic field (one induces the other according to Maxwell's equations) in free space.

b) Taking the general case of $\overline{E} = E_x\overline{a_x} + E_y\overline{a_y} + E_z\overline{a_z}$ as all components have to conform to the wave equations, a general solution will have the form:

$$E_x\overline{a_x} = E_{xo}e^{\pm j(\omega t - \beta z)}\overline{a_x} + E_{xo}e^{\pm j(\omega t - \beta y)}\overline{a_x} + E_{xo}e^{\pm j(\omega t - \beta x)}\overline{a_x}$$

(as we are considering propagation in free space we do not need to consider any waves travelling in the -*x* direction as there are no boundaries to reflect from).

Therefore substituting any given component into the wave equation one gets

$\beta^2 = \epsilon_o\, \mu_o \omega^2$ and $\frac{\omega}{\beta} = 1/\sqrt{\epsilon_o\, \mu_o}$. The velocity of any wave component is obtained by considering an observer travelling at the same speed as the wave. In which case the value of the electric (magnetic) field will be constant. Hence $(\omega t - \beta_x x) = constant$

$\frac{\partial}{\partial t}(\omega t - \beta_x x) = 0;\ \frac{\partial x}{\partial t} = \frac{\omega}{\beta_x}$ is the velocity of the electromagnetic wave in the x direction.

But form the general solution we see that this is a constant, which is the same in any direction.

Using the data book values for ϵ_o and μ_o,

$\frac{1}{\sqrt{\epsilon_o\,\mu_o}} = \frac{1}{\sqrt{8.854x10^{-12}.4\pi x10^{-7}}} = 2.998\;x\;10^8\;ms^{-1}$. This is the same as the measured velocity of visible light (see data book for velocity of light). Hence the conclusion that visible light is a form of electromagnetic wave, with electric and magnetic field components.

c) Considering the Maxwell equation $\overline{\nabla}\,X\overline{E} = -\mu_o \frac{\partial}{\partial t}\overline{H}$

$$\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\bar{a}_x + \left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)\bar{a}_y + \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\bar{a}_z = -\mu_o\frac{\partial}{\partial t}(H_x\overline{a_x}+H_y\overline{a_y}+H_z\overline{a_z})$$

As $\overline{E} = E_x\overline{a_x}$ as it is confined in the x-y plane and defined only in one direction, and it does not vary in the x-y plane $\frac{\partial E_x}{\partial y} = \frac{\partial E_x}{\partial x} = 0$,

$$\frac{\partial E_x}{\partial z}\bar{a}_y = -\mu_o\frac{\partial}{\partial t}(\overline{H_y a_y})$$

And

$$-\frac{\partial H_y}{\partial z}\bar{a}_x = \epsilon_o\frac{\partial}{\partial t}(\overline{E_x a_x})$$

From the general solution above with only one component of $\overline{E}$ and $\overline{H}$

For a plane wave one gets

$$\frac{E_x}{H_y} = \frac{\omega\mu_o}{\beta}$$

From the solution to b) we know $\frac{\omega}{\beta} = 1/\sqrt{\epsilon_o\,\mu_o}$

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu_o}{\epsilon_o}} = 377\;\Omega$$

Which has units of ohms (Vm^{-1}/ Am^{-1}) and is defined as the characteristic impedance of free space. It allows one to express the magnitude of magnetic field as a constant factor of the orthogonal electric field magnitude and vice versa.

d)

(i) As the E field is given by $\overline{E} = (E_o\sin\theta\,\sin(\omega t - 2\pi r/\lambda))/r.\bar{e}_\theta$ the corresponding magnetic field is given by

$\overline{H} = (E_o\sin\theta\,\sin\omega(t - 2\pi r/\lambda))/(r\sqrt{\frac{\mu_o}{\epsilon_o}}).\bar{e}_\phi$ The orientation in the ϕ where as the electric field is in the θ direction.

(ii) From Poynting's theorem for power density in the field

$$\bar{S} = \overline{E}X\overline{H}$$

$\bar{S} = (E_o^2 sin^2\theta\, sin^2(\omega t - 2\pi r/\,\lambda))/(r^2\sqrt{\frac{\mu_o}{\epsilon_o}}).\bar{e}_r$ Wm^{-2}

Power transmission is in the radial direction orthogonal to θ and ϕ.

(iii) The power density at r = 100λ = R will be

$$\bar{S} = (E_o^2 sin^2\,\theta\, sin^2(\,\omega t - 2\pi 100))/(R^2\sqrt{\frac{\mu_o}{\epsilon_o}}) \,.\, \bar{e}_r$$

Additionally θ = π/2.

The area over which the receiving antenna collects power is $R\delta\theta . R\sin\theta\delta\emptyset$

Hence $\quad Power\ to\ antenna = E_o^2\, sin^2(\omega t - 2\pi 100))R^2\delta\theta\delta\emptyset/(R^2\sqrt{\frac{\mu_o}{\epsilon_o}})\,.$

Average $Power\ to\ antenna = \frac{1}{2\pi}E_o^2 . \delta\theta\delta\emptyset/\sqrt{\frac{\mu_o}{\epsilon_o}} . \int_0^{2\pi} sin^2(\,\omega t - 2\pi 100)d\omega t$

$$= \frac{800x0.01}{377} = 21mW$$

($sin^2\theta = 0.5(1 - \cos 2\theta)$ and integral of cos 2θ is 0 over the integration of one period)

A question on Maxwells's equations and free space radiation. This was the more popular question in Section C. The first part (a – c) concerned the prediction of EM waves in free space from Maxwell's equations. It was pleasing (especially given this is perhaps the most demanding material encountered in 1B but presented at the very end) that a significant majority of the candidates were proficient in vector analysis to use Maxwell's equations to predict wave propagation. They were also aware of the physical consequences of this prediction. The last part was a problem based around transmitting and receiving radio antennae. Again, it was pleasing, in fact surprisingly so, how a large number of candidates understood the concepts of EM radio transmission and were able to handle the analysis. Many got perfect answers, to what was a challenging problem. It is reassuring to know that we have some very able and excellent students. Unfortunately, the average mark does not reflect this as the not so good 20% scored less than 30%!