Part IB Paper 6 - Solutions - June 2015

1. (a) From fig. 2, the phase reaches $-180°$ when $\omega \simeq 1.7$ rad/s.
At this point $|G(j\omega)| = 23\,dB = 14$
For stability $|k_p G(j\omega)| < 1$ at $\omega = 1.7$
$\therefore\ k_p < \frac{1}{14} = 0.07$
The phase never quite reaches $-180°$ below $\omega = 1.7$
so $\underline{0 \leq k_p < \frac{1}{14} = 0.07}$ for stability.

(b) On the magnitude Bode plot, the gradient of the straight parts of the curve is approx $-40\,dB$/decade, and the phase shift at $\omega = 10^{-1}$ and $\omega = 10$ rad/s. is approx $-180°$.
Hence $\underline{n = -2}$, since $G(j\omega) \propto \frac{1}{(j\omega)^2}$ at very low and very high frequencies.

There is a notch (inverted peak) at 1 rad/s and a peak at 1.4 rad/s in the magnitude plot, and the phase $\simeq -90°$ at these two points.
Hence $a = T_n^2$, where T_n is the time const. of the numerator, and $T_n = 1$s. so that $\omega T_n = 1$ at the frequency of the notch. $\therefore\ \underline{a = T_n^2 = 1}$

Similarly $c = T_d^2$, where T_d is the time const. of the denominator, and $T_d = \frac{1}{1.4}$ s. so that $\omega T_d = 1$ at the frequency of the peak. $\therefore$ $\underline{\underline{c = T_d^2 = \left(\frac{1}{1.4}\right)^2 = \frac{1}{2}}}$.

To find b & d, we must solve for the damping factors ξ_n & ξ_d. Draw straight line asymptotes, with slopes −40 dB/decade as shown in Fig. 2. The notch is approx 8 dB below its asymptote & the gain of the numerator term is $2\xi_n$ at this point.

$\therefore -8\,dB = 0.40 = 2\xi_n \quad \therefore \xi_n = 0.2$ & $b = 2\xi_n T_n$

$\underline{\underline{b = 0.4}}$

The peak is approx 17 dB above its asymptote and the gain of the denominator term is $1/2\xi_d$ at this point.

$\therefore 17\,dB = 7 = \frac{1}{2\xi_d} \quad \therefore \xi_d = \frac{1}{14} = 0.07$

and $\underline{\underline{d}} = 2\xi_d T_d = 2 . 0.07 . \frac{1}{1.4} \underline{\underline{= 0.1}}$

Finally to find k, we choose $\omega = 0.1$ rad/s and observe that

$$|G(j\omega)| \simeq k.(0.1)^{-2}.\frac{1}{1} = 1000 \quad (60\,dB)$$

$$\therefore \underline{\underline{k}} = (0.1)^2 . 1000 \underline{\underline{= 10}}$$

$\therefore a = 1,\ b = 0.4,\ c = 0.5,\ d = 0.1,\ k = 10,\ n = -2.$

1. (c) For $K(j\omega) = \dfrac{j\omega + 1}{j\omega + 4}$

At $\omega = 1$: $|K(j1)| = \sqrt{\dfrac{1+1}{1+16}} \sim -9\,\text{dB}$

$\angle K(j1) = \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{4}\right) = 45^\circ - 14^\circ = 31^\circ$

At $\omega = 2$: $\angle K(j2) = \tan^{-1}\left(\frac{2}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 63.5^\circ - 26.5^\circ = 37^\circ$

At $\omega = 4$: $|K(j4)| = \sqrt{\dfrac{16+1}{16+16}} \sim -3\,\text{dB}$

$\angle K(j4) = \tan^{-1}\left(\frac{4}{1}\right) - \tan^{-1}\left(\frac{1}{1}\right) = 76^\circ - 45^\circ = 31^\circ$

Hence $K(j\omega)$ plot is as shown on fig 2.

The phase margin is measured at the frequency where the return ratio is unit magnitude.

Hence we require $|K(j\omega).G(j\omega)| = 1 = 0\,\text{dB}$.

From fig. 2 we see that at $\omega = 4$ rad/s, ~~$|K(j4)| = -3\,\text{dB}$~~ $|K(j4)| \simeq -3\,\text{dB}$ and $|G(j4)| \simeq +3\,\text{dB}$.

$\therefore$ $\omega = 4$ rad/s gives $|K(j\omega).G(j\omega)| = 0\,\text{dB}$.

Hence the phase margin is $\angle K(j4) + \angle G(j4) + 180^\circ$

$= 31^\circ - 182^\circ + 180^\circ = \underline{\underline{29^\circ}}$

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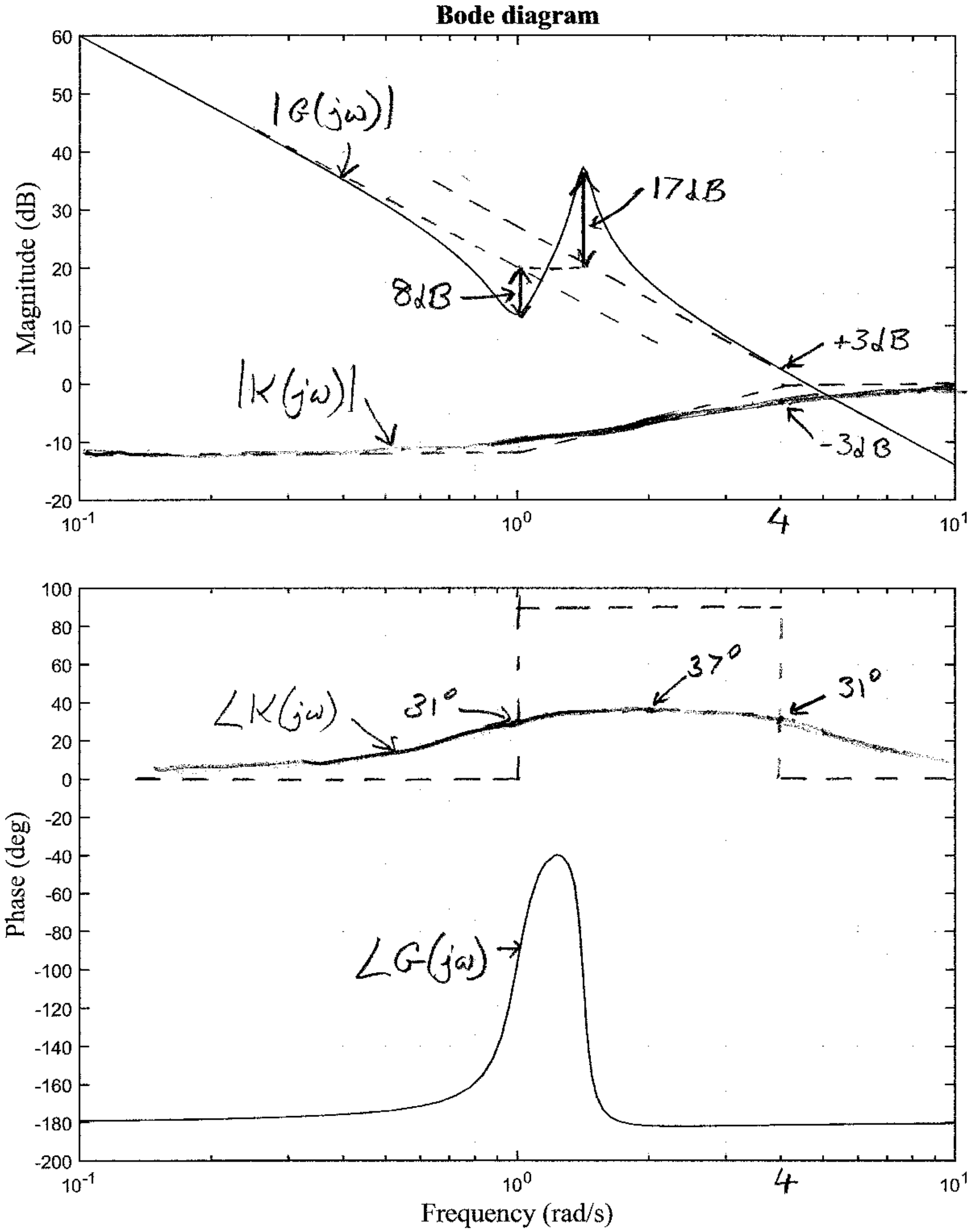
EGT1
ENGINEERING TRIPOS PART IB
Thursday 4 June 2015, Paper 6, Question 1.

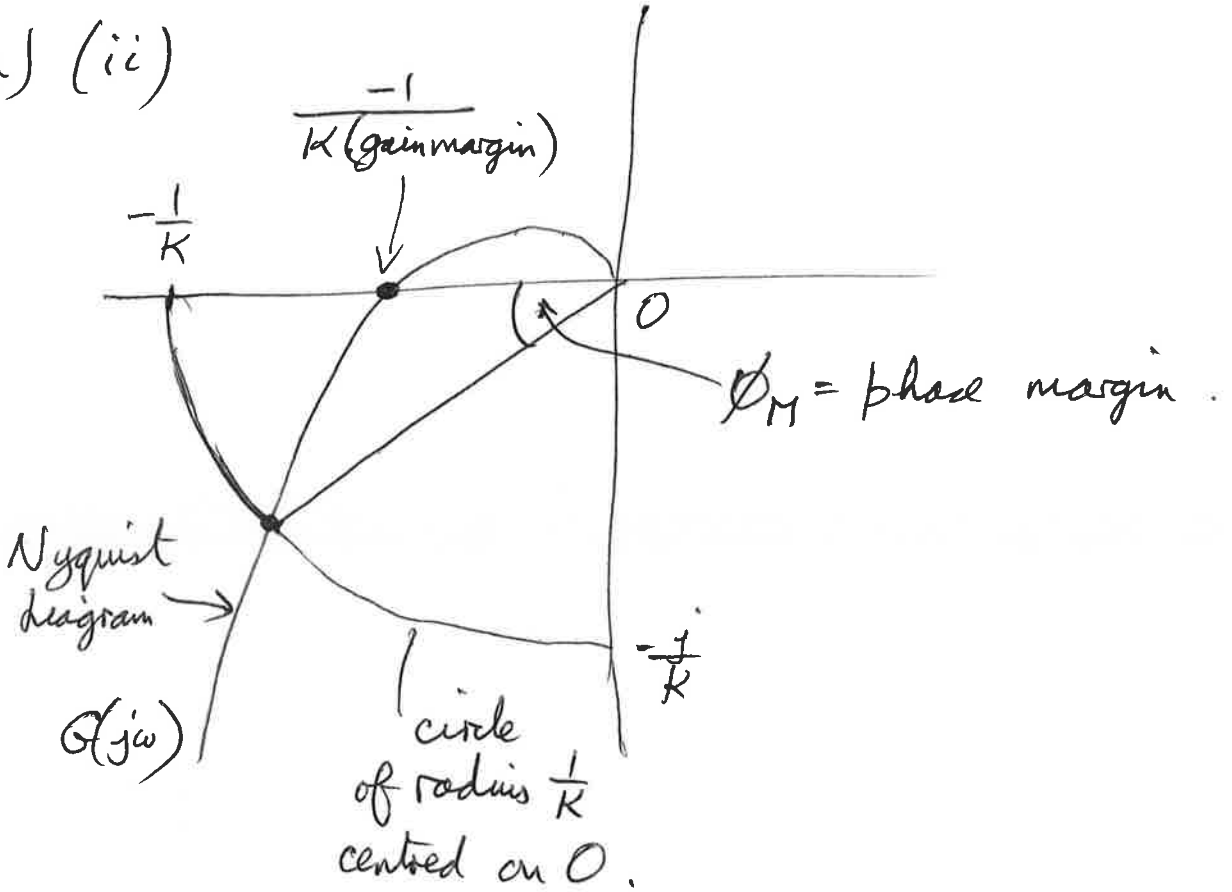


Extra copy of Fig. 2: Bode diagram for Q1.

2. (a) (i) To obtain Nyquist diagram of a system, we need to measure $|G(j\omega)|$ and $\angle G(j\omega)$ as ω increases from zero to a frequency much greater than the point where $|G| = 1$. We need to excite the system with a sinusoid of suitable magnitude (preferably unity) and measure the output magnitude and phase shift from input to output as ω is increased, as a function of ω. ~~The~~ $|G(j\omega)|$ is the ratio of output amplitude to input amplitude at each frequency. $\angle G(j\omega)$ is the phase shift. Care must be taken to avoid overload or non-linearities in the system, and also to keep the signals large enough to avoid poor signal-to-noise ratios.

To plot the Nyquist diagram, at each frequency we plot $G(j\omega)$ as a complex vector from the origin and join up all the tips of the vectors into a smooth curve.

2. (a) (ii)



Gain margin is the amount of extra gain that is needed for the Nyquist plot to pass through the point $\frac{-1}{K}$.

Phase margin is the amount of additional negative phase shift needed for the Nyquist plot to pass through $\frac{-1}{K}$.

2 (b)(i) The Nyquist plot of $G(j\omega)$ in fig. 4 crosses the negative real axis at $(-0.2, 0)$, so the system will be stable up to $k_p = \frac{1}{0.2} = 5$.

Hence it is stable for $\underline{\underline{0 \leq k_p < 5}}$

so that the point $\frac{-1}{k_p}$ will always lie to the left of the point $(-0.2, 0)$.

(ii) The phase margin is maximised when the vector from the origin to the phase-margin point is at its largest angle from the negative real direction, and will be at the cusp of the plot at $\omega = 1.4$ rad/s. This cusp is 0.46 units from the origin so $\frac{1}{k_p} = 0.46$

$\therefore \underline{k_p = 2.17}$ to maximise the phase margin.

(iii) If $r(t) = \cos(2.5t)$, then $\omega = 2.5$ rad/s.

$\therefore G(j.2.5) = -0.28 - 0.07j$ from fig. 4.

Hence $$\frac{\bar{y}}{\bar{r}} = \frac{k_p G(j.2.5)}{1 + k_p G(j.2.5)} = \frac{2(-0.28 - 0.07j)}{1 + 2(-0.28 - 0.07j)}$$

$$= 1.25 \angle -2.59 \text{ rad} = 1.25 \angle -148^\circ$$

$\therefore$ when $r(t) = \cos(2.5t)$,

$$y(t) = \underline{\underline{1.25 \cos(2.5t - 2.59)}} \text{ in steady state.}$$

2(b)(iv) The sensitivity function is

$$\left| \frac{1}{1 + \text{return ratio}} \right| = \left| \frac{1}{1 + k_p G(j\omega)} \right|$$

$$= \left| \frac{1/k_p}{1/k_p + G(j\omega)} \right| \qquad \text{and} \quad k_p = 2 .$$

The circle centred at $\frac{-1}{k_p}$ and of radius $\frac{1}{k_p}$ intersects the plot of $G(j\omega)$ at $\omega \simeq 0.68$ rad/s. and nowhere else except at the origin.

Hence $\left| 1/k_p + G(j\omega) \right| > \left| 1/k_p \right|$ to keep the sensitivity function < 1 for $\underline{\underline{0 \leq \omega < 0.68}}$ rad/s.

To maximise the sensitivity function, $\left| 1 + k_p G(j\omega) \right|$ must be minimised. This occurs when $G(j\omega)$ is as close to $-1/k_p$ as it can be, and is when $\underline{\underline{\omega \simeq 2.4}}$ rad/s.

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EGT1
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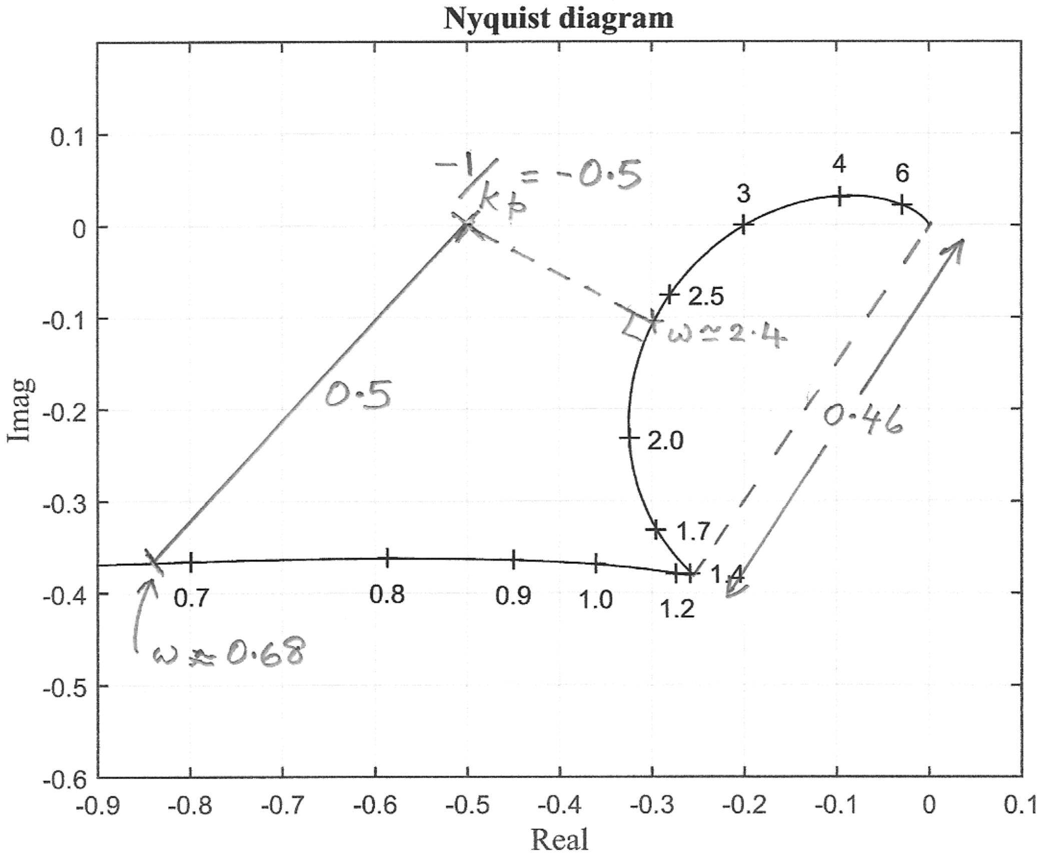
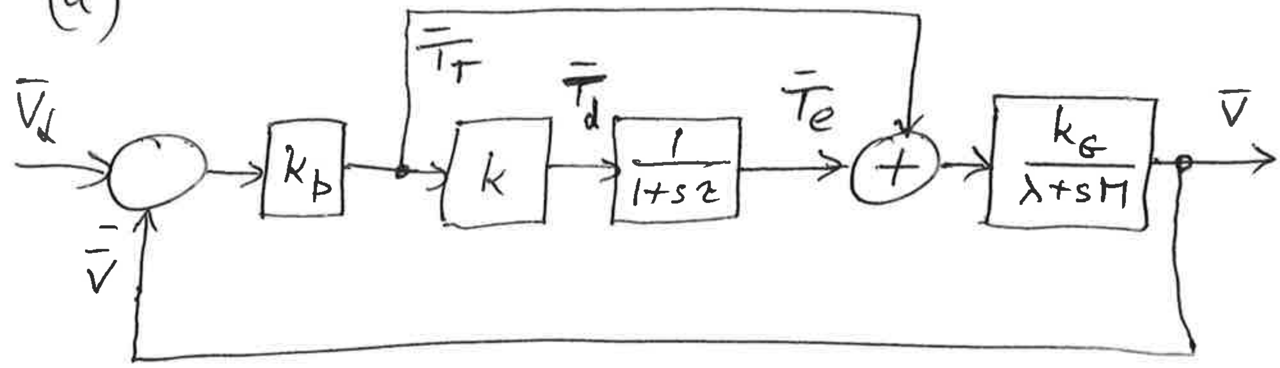
Thursday 4 June 2015, Paper 6, Question 2.



Extra copy of Fig. 4: Nyquist diagram for question 2.

3. (a)



$$\bar{V} = (\bar{V}_d - \bar{V}) \cdot G(s)$$

where $G(s) = k_p\left(1 + \frac{k}{1+s\tau}\right) \cdot \frac{k_G}{\lambda + sM}$

$$\therefore \quad \frac{\bar{V}}{\bar{V}_d} = \frac{G(s)}{1+G(s)} = \frac{k_p(1+s\tau+k) \cdot k_G}{(1+s\tau)(\lambda+sM) + k_p(1+s\tau+k)k_G}$$

(b) Expanding the denominator of $\frac{\bar{V}}{\bar{V}_d}$ gives:

$$\lambda + s(\lambda\tau + M) + s^2\tau M + k_p k_G(1 + k + s\tau)$$

$$= \lambda + k_p k_G(1+k) + s(\lambda\tau + M + k_p k_G \tau) + s^2 \tau M$$

Equating this to $C + sB + s^2A = C(1 + 2\xi sT + s^2T^2)$

we see that $T = \sqrt{\frac{A}{C}}$ & $\xi = \frac{B}{2TC} = \frac{B}{\sqrt{4AC}}$

Putting in values: $C = \lambda + k_p k_G(1+k) = 2.5 + 50(1+k)$

$B = (\lambda + k_p k_G)\tau + M = 105 + 80 = 185$

$A = \tau M = 160$

If $\xi \geq 0.7 \qquad B^2 \geq (0.7)^2 \cdot 4AC = 2AC$

3(b) - cont. $\therefore \quad C \leqslant \frac{B^2}{2A}$

ie $\quad 52.5 + 50k \leqslant \frac{185^2}{2.160} = 106.95$

$$\therefore 50k \leqslant 54.45$$

$$\therefore k \leqslant \underline{\underline{1.09}}$$

When $s=0$, $\quad \frac{\bar{V}}{\bar{V}_d} = \frac{k_p k_G (1+k)}{\lambda + k_p k_G (1+k)}$

$$= \frac{50(1+k)}{2.5 + 50(1+k)}$$

$$= \frac{50.\,2.09}{2.5 + 50.\,2.09} = 0.97\cancel{4}66$$

$\therefore \quad \bar{V}_d - \bar{V} = \bar{V}_d\left(1 - \frac{\bar{V}}{\bar{V}_d}\right) = \bar{V}_d\left(1 - \overset{0.9766}{\cancel{0.9766}}\right) = 0.0234\,\bar{V}_d$

$\therefore$ S.S. error $= 0.0234 \times 6 = \underline{\underline{0.14}}$ m/s.

3(c) With the new controller $\frac{k}{1+s\tau}$ is replaced by:

$$2\cancel{3}.\,\frac{1+s\tau}{1+s(\tau/4)}.\,\frac{1}{1+s\tau} = \frac{\cancel{3}2}{1+s.\,\tau/4}$$

Hence we have k replaced by $\cancel{3}2$ and τ is replaced by $\tau/4$.

New damping ratio $= \frac{B}{\sqrt{4AC}}$ where the new values for A, B, C are:

$C = 2.5 + 50(1+2) = \cancel{202.5}\ 152.5$

$B = (2.5+50).\cancel{3}\,\frac{2}{4} + 80 = 26.25 + 80 = 106.25$

$A = \frac{2}{4}.\,80 = 40.$ $\qquad \therefore \xi_{new} = \frac{106.25}{\sqrt{4.\,40.\,\cancel{202.5}\,152.5}} = 0.680$

3(c) - cont.

The new steady state gain $\frac{V}{V_d}$ will be given by

$$\frac{50.3}{2.5 + 50.3} = 0.9836$$

$\therefore$ SS error $= (1 - 0.9836) \times 6 = \underline{\underline{0.098}}$ m/s

(d) The damping is almost the same, but there two key ~~changes~~ differences:

1. The power-assist gain effectively increases from 1.09 to 2, so that the rider will now need to supply only $\frac{1}{3}$ of the motive ~~T~~ torque, instead of almost $\frac{1}{2}$ of the torque previously.

2. The time constant $\tau = 2s$ ~~as~~ is replaced by $\frac{\tau}{4} = \frac{1}{2}s$. so that the response of the power-assist will be quicker.

4 (a) From data book: $P(\omega) = \int_{-\infty}^{\infty} p(t) e^{-j\omega t} dt$

Hence the F.T. of $p(t) e^{j\omega_0 t}$ is given by

$$\int_{-\infty}^{\infty} p(t) e^{j\omega_0 t} \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} p(t) e^{-j(\omega-\omega_0)t} dt$$
$$= P(\omega-\omega_0)$$

$\therefore P(\omega-\omega_0)$ is the F.T. of $p e^{j\omega_0 t}$.

(b) $q(t) = p(t)(1+\cos\omega_0 t)$

$$= p(t)\left(1 + \tfrac{1}{2} e^{j\omega_0 t} + \tfrac{1}{2} e^{-j\omega_0 t}\right)$$

$\therefore Q(\omega) = P(\omega) + \cancel{\tfrac{1}{2}Q(\omega)} \tfrac{1}{2}P(\omega-\omega_0) + \tfrac{1}{2}P(\omega+\omega_0)$

from the result in (a)

From the databook: $P(\omega) = ab \operatorname{sinc}\left(\frac{\omega b}{2}\right)$

where $a=1$, $b = \frac{2\pi}{\omega_0} = 2T$

$$P(\omega) = 2T \operatorname{sinc}(\omega T)$$

$\therefore Q(\omega) = T\left[\operatorname{sinc}(\omega-\omega_0)T + 2\operatorname{sinc}(\omega T) + \operatorname{sinc}(\omega+\omega_0)T\right]$

(c) When $\omega = \frac{\pi}{T}\left\{0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, 2\tfrac{1}{2}, 3\right\}$,

$$\omega T = 0, \tfrac{\pi}{2}, \pi, \tfrac{3\pi}{2}, 2\pi, \tfrac{5\pi}{2}, 3\pi \ldots$$

Hence

$\operatorname{sinc}(0) = 1$, $\operatorname{sinc}\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{2/\pi} = \frac{2}{\pi}$, $\operatorname{sinc}(\pi) = 0$

$\operatorname{sinc}\left(\frac{3\pi}{2}\right) = \frac{-1}{2/3\pi} = \frac{-2}{3\pi}$, $\sin(2\pi) = 0$, $\operatorname{sinc}\left(\frac{5\pi}{2}\right) = \frac{2}{5\pi}$

$\operatorname{sinc}(3\pi) = 0$. $\operatorname{sinc}(.)$ is even so these will apply for negative arguments too.

$\text{sinc}\,(\omega \pm \omega_0)T = \text{sinc}\,(\omega T \pm \pi)$

So $\text{sinc}(\omega-\omega_0)T$ and $\text{sinc}(\omega+\omega_0)T$ will just be $\text{sinc}(\omega T)$ shifted left and right by π. Hence ~~a sketch of the 3 sincs~~ $Q(\omega)$ will be given by:

$$Q(0) = T\{\text{sinc}(-\pi) + 2\,\text{sinc}(0) + \text{sinc}(\pi)\}$$
$$= T\{0+2+0\} = 2T$$

$$Q\left(\tfrac{\pi}{2}\right) = T\left\{\text{sinc}\left(\tfrac{-\pi}{2}\right) + 2\,\text{sinc}\left(\tfrac{\pi}{2}\right) + \text{sinc}\left(\tfrac{3\pi}{2}\right)\right\}$$
$$= T\left\{\frac{2}{\pi} + \frac{4}{\pi} - \frac{2}{3\pi}\right\} = \frac{16}{3\pi}T = 1.698\,T$$

$$Q(\pi) = T\{\text{sinc}(0) + 2\,\text{sinc}(\pi) + \text{sinc}(2\pi)\}$$
$$= T\{1+0+0\} = T$$

$$Q\left(\tfrac{3\pi}{2}\right) = T\left\{\text{sinc}\left(\tfrac{\pi}{2}\right) + 2\,\text{sinc}\left(\tfrac{3\pi}{2}\right) + \text{sinc}\left(\tfrac{5\pi}{2}\right)\right\}$$
$$= T\left\{\frac{2}{\pi} - \frac{4}{3\pi} + \frac{2}{5\pi}\right\} = \frac{16}{15\pi}T$$
$$= \cancel{0.255}\ 0.340\,T$$

$$Q(2\pi) = T\{\text{sinc}(\pi) + 2\,\text{sinc}(2\pi) + \text{sinc}(3\pi)\}$$
$$= T\{0+0+0\} = 0$$

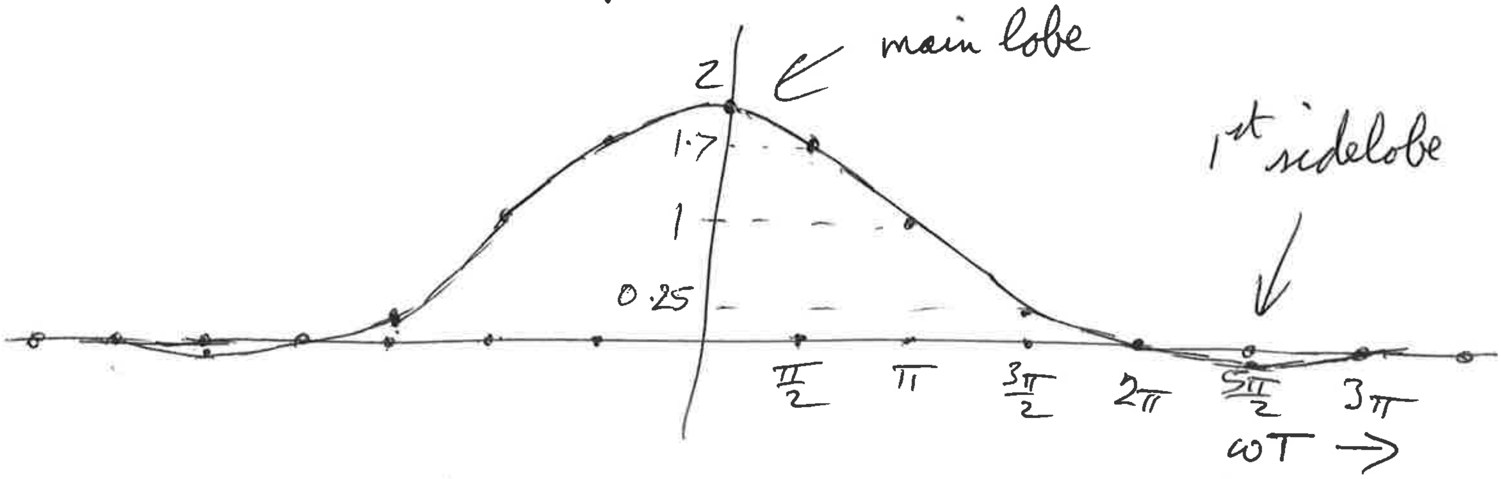
$$Q\left(\tfrac{5\pi}{2}\right) = T\left\{\text{sinc}(3\pi/2) + 2\,\text{sinc}\left(\tfrac{5\pi}{2}\right) + \text{sinc}\left(\tfrac{7\pi}{2}\right)\right\}$$
$$= T\left\{\frac{-2}{3\pi} + \frac{4}{5\pi} - \frac{2}{7\pi}\right\}$$
$$= T\{-0.2122 + 0.2546 - 0.0909\} = -0.0485\,T$$

$$Q(3\pi) = T\{0+0+0\} = 0$$

Because $\text{sinc}(\cdot)$ is even, $Q(\omega)$ will also be even.

4(b) - cont.

Hence a sketch of $Q(\omega)$ is :



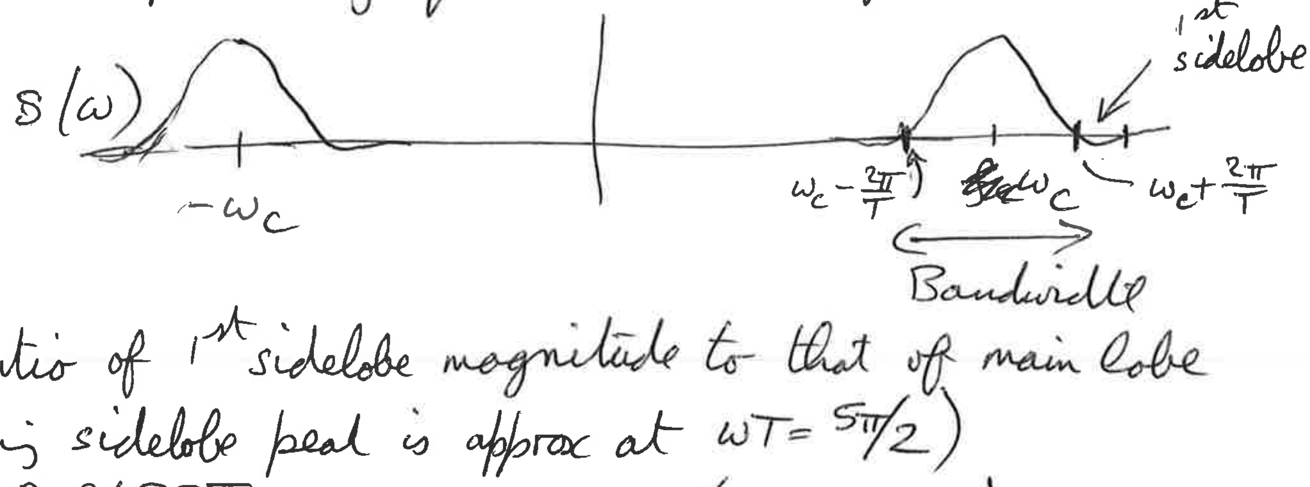
(d) If $q(t)$ is modulated onto a carrier by $s(t) = q(t)\cos(\omega_c t) = q(t).\frac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)$

then $S(\omega) = \frac{1}{2}Q(\omega-\omega_c) + \frac{1}{2}Q(\omega+\omega_c)$ from (a).

Hence $Q(\omega)$ will be shifted to $\pm\omega_c$.

The first zeros of $Q(\omega)$ are at $\pm 2\pi/T$, so the bandwidth of $Q(\omega-\omega_c)$ is $\underline{4\pi/T \text{ rad/s}}$ or $\underline{2/T \text{ Hz}}$.

This is also the bandwidth of $S(\omega)$, since we only look at positive frequencies when a signal is bandpass.



Ratio of 1st sidelobe magnitude to that of main lobe (assuming sidelobe peak is approx at $\omega T = 5\pi/2$)

$$= \frac{0.0485T}{2T} = \underline{0.0243} \quad (-32.3 \text{ dB}).$$

5. (a) (i) If $x_n = e^{j\frac{2\pi k}{N} n}$

then $X_m = \sum_{n=0}^{N-1} e^{j\frac{2\pi(k-m)n}{N}}$

This will sum to zero, <u>except</u> when $k=m$, when it will sum to N. (Scaled up by N)

Hence X_m represents the amplitude of the component at frequency $\frac{2\pi m}{N}$ times sampling freq.

~~But~~ $\cancel{e^{j\frac{2\pi(k-m)n}{N}}}$ Hence with a sampling period of $100\mu s$, sampling freq $= 10^4$ Hz & the freq measured by X_m is $\frac{2\pi m}{N} \cdot 10^4$ rad/s $= \underline{\frac{m}{N} \cdot 10^4 \text{ Hz}}$

But $X_{m+N} = X_m$

since $e^{j\frac{2\pi(k-m-N)n}{N}} = e^{j\frac{2\pi(k-m)n}{N}} \cdot e^{-j2\pi n}$

$= e^{j\frac{2\pi(k-m)n}{N}}$

The applies for all m, and so X is periodic over N samples, since $X_m = X_{m+N} = X_{m+2N} \cdots$

(a) (ii) If $x = \{0, 1, 1, 0, -1, -1\}$ & $N=6$

$X_m = e^{-j\frac{2\pi m}{N}} + e^{-j\frac{2\pi \cdot 2m}{N}} - e^{-j\frac{2\pi \cdot 4m}{N}} - e^{-j\frac{2\pi \cdot 5m}{N}}$

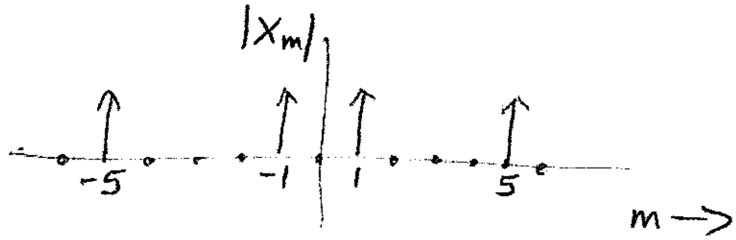
$= e^{-j\frac{2\pi m}{N}} + e^{-j\frac{2\pi 2m}{N}} - e^{-j\frac{2\pi(-2m)}{N}} - e^{-j\frac{2\pi(-m)}{N}}$

(since x is assumed periodic over 6 samples)

$= -2j \sin\left(\frac{2\pi m}{6}\right) - 2j \sin\left(\frac{4\pi m}{6}\right)$

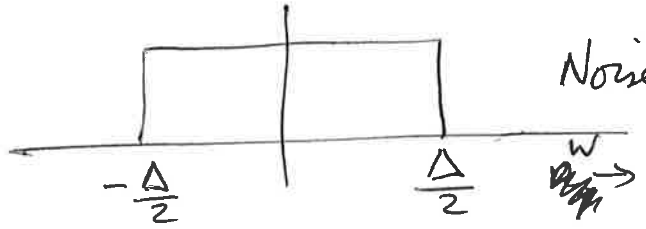
Substituting $m = 0$ to 5

$$X_0 = 0$$
$$X_1 = -2j\left[\sin\left(\tfrac{\pi}{3}\right) + \sin\left(\tfrac{2\pi}{3}\right)\right] = -2j\,.\sqrt{3}$$
$$X_2 = -2j\left[\sin\left(\tfrac{2\pi}{3}\right) + \sin\left(\tfrac{4\pi}{3}\right)\right] = -j\left[\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}\right] = 0$$
$$X_3 = -2j\left[\sin(\pi) + \sin(2\pi)\right] = 0$$
$$X_4 = X_{-2} = 0$$
$$X_5 = X_{-1} = -2j\left[\sin\left(\tfrac{-\pi}{3}\right) + \sin\left(\tfrac{-2\pi}{3}\right)\right] = +2j\sqrt{3}$$



The spectrum is periodic so X_1 & X_5 correspond also to X_1 & X_{-1}, which are the positive and negative components of a single sinusoid, $A\sin\left(\frac{2n\pi}{6}\right)$. In fact the given samples of x correspond to those of a single sinusoid sampled at $\left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ and multiplied by $\frac{2}{\sqrt{3}}$. The other spectral terms are zero because the input is just a single sinusoid at $\frac{1}{6}$ of the sampling frequency.

5 (b) (i)



The quantised signal x_q may be modelled as

$$x_q = x + w$$

where w is error noise with a rectangular pdf as shown above, where Δ is the quantiser step size.

For the given quantiser, $\Delta = \frac{2V}{2^n} = 2^{1-n}\,V.$

The mean square error is $\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} w^2 . f(w)\, dw$

where $f(w)$ is the pdf of the noise and

$$f(w) = \begin{cases} 1/\Delta & \text{if } |w| < \frac{\Delta}{2} \\ 0 & \text{elsewhere} \end{cases}$$

since $f(w)$ must integrate to unity to be a valid pdf.

$$\therefore \text{MSE} = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{w^2}{\Delta}\, dw = \left[\frac{w^3}{3\Delta}\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = 2 . \frac{(\Delta/2)^3}{3\Delta} = \frac{\Delta^2}{12}$$

$$\therefore \text{RMS quantising error} = \sqrt{\frac{\Delta^2}{12}} = \frac{\Delta}{\sqrt{12}} = \frac{2^{1-n}}{\sqrt{12}}\ V$$

$$= \underline{\underline{\frac{2^{-n}}{\sqrt{3}}\ V.}}$$

For this to be $< 10^{-3}$ V, $2^{-n} < 10^{-3} . \sqrt{3}$

$$\therefore 2^n > \frac{1000}{\sqrt{3}}$$

$$\therefore \underline{\underline{n = 10 \text{ bits}}}$$

5 (b) (ii)

Nyquist's sampling theorem states that the sampling rate must be twice the signal bandwidth. In practice, we must allow an extra 10% to 20% of extra rate because the anti-aliasing filters are non-ideal and need finite transition bands. Hence sampling rate must be $(2 \times 20\text{kHz}) \times 1.1$ or 1.2, giving 44 to 48 kHz. We will assume 44 kHz. With 10 bits per sample, the bit rate will then be 440 kbit/s. (A stereo audio signal would need twice this bit rate.)

(iii) If 5 parity check bits are needed as well as 10 source data bits, the block size will be 15 bits per audio sample, so the bit rate will increase to $44 \times 15 = 660$ kbit/s.

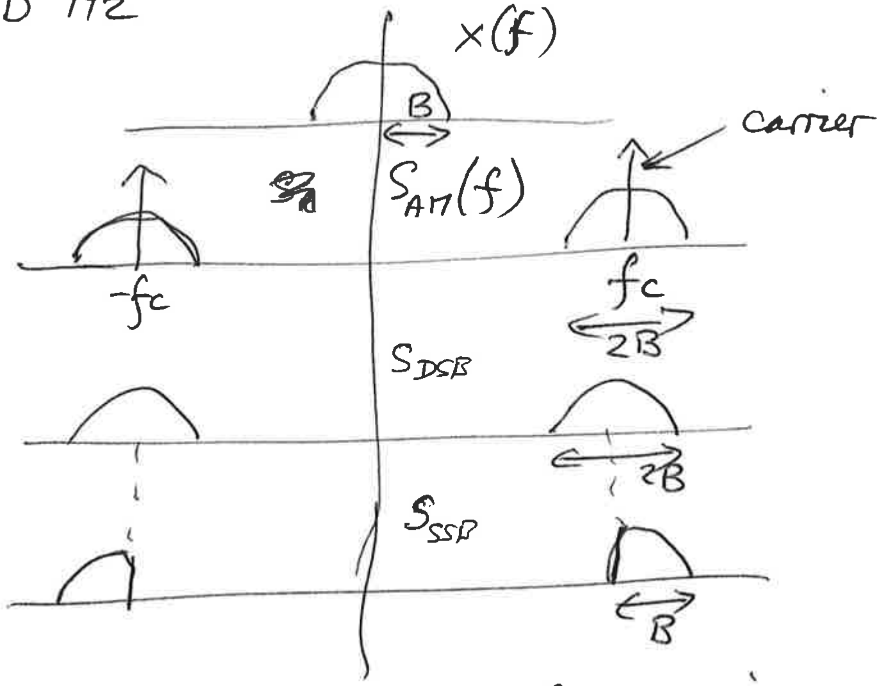
Block codes like this can correct errors when they occur on the link, and help to maintain good audio quality under poor channel signal-to-noise conditions.

[Note: this code would probably be able to ~~detect~~ correct a single error per block and detect double errors per block.]

6 (a) AM: $s_{AM}(t) = (a_0 + x(t))\cos(2\pi f_c t)$

Spectrum of AM: $S_{AM}(f) = \frac{a_0}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right)$
$$+ \frac{1}{2}\left[X(f-f_c) + X(f+f_c)\right]$$

If baseband bandwidth of ~~X(f)~~ (from zero freq) is B Hz, then the bandwidth of $S_{AM}(f)$ about each carrier f_c is $2B$ Hz



DSB-SC modulation is AM without the carrier components $\delta(f-f_c)$ & $\delta(f+f_c)$. It has the same BW as AM, but needs less transmit power.

SSB-SC modulation is DSB-SC with one sideband removed. It has ~~the same~~ half the bandwidth of DSB & AM, but needs the same power as DSB.

FM needs much more bandwidth than AM but is better at rejecting channel noise, & so gives a better SNR at the receiver than AM.

6 (b) PAM comprises a mapping from bits to real or complex ~~numbers~~ symbols, eg $0 \to -A$, $1 \to +A$; followed by convolution of the mapped ~~values~~ symbols with a shaping pulse $p(t)$, which is usually time limited to approximately the symbol period.

The pulse $p(t)$ should have the following properties:

1. Time limited to approx T, the symbol period so that detection of one pulse is not affected by the previous or subsequent pulses.
2. Bandlimited so that the spectrum of the signal $\sum_k X_k \, p(t-kT)$ is also bandlimited. For random symbols X_k we find that the signal spectrum is the same as that of $p(t)$.

(c) If the constellation ~~poin~~ symbols X_k are complex then we can create a modulated signal from $x(t) = Re\left[x_b(t) \, e^{j2\pi f_c t} \right]$

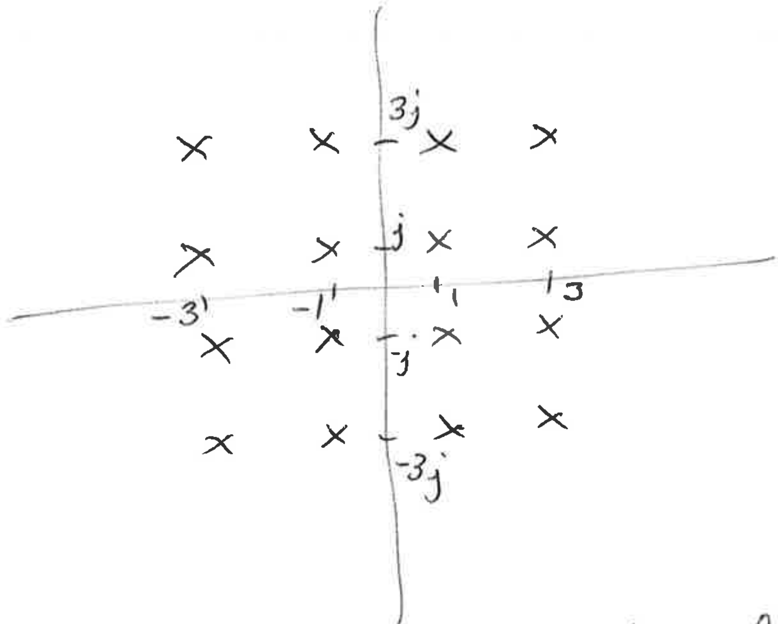
where $x_b(t) = \sum_k X_k \, p(t-kT)$

Hence $x(t) = Re(x_b) \cos(2\pi f_c t) - Im(x_b) \sin(2\pi f_c t)$

$$= \sum_k p(t-kT) \left[Re[X_k] \cos(2\pi f_c t) - Im[X_k] \sin(2\pi f_c t) \right]$$

6(c) - cont.

A 16-symbol QAM constellation is



Hence ~~?~~ $Re[X_k]$ can take on 4 values:

$-3A, -A, A, 3A$

& $Im[X_k]$ can also take on ~~th~~ same set of values, but coding different bits from the data.

6(d) The rule for ~~of~~ optimally detecting equi-probable symbols in noise, is to choose the symbol ~~nearest~~ constellation point ('x' in the above diagram) nearest to the received signal value. Hence ~~the~~ for 16-QAM the region boundaries are the lines passing horizontally and vertically mid-way between the points

Y_k-space

square region of received Y_k values that will be detected as the symbol in the centre of the region.

X_k

6 (e) Each symbol of M states can represent ~~2^m~~ m bits of information where $M = 2^m$ (or $m = \log_2 M$). Hence the data rate, which equals m times the symbol rate, increases in proportion to m. This increases quite slowly with M, due to the $\log_2(\cdot)$ relationship.

The linear size of the regions of V_k tend to decrease with increasing M, in proportion to $\frac{1}{\sqrt{M}}$ if the mean squared radius of the constellation stays approx. constant. Hence the bit error probability will increase and the link performance will degrade quite rapidly with increasing M.

Therefore the choice of M is usually a tradeoff between increasing data rate (good) and greater noise sensitivity (bad).