EGT1
ENGINEERING TRIPOS PART IB

Thursday 6 June 2019 2 to 4.10

Paper 6

INFORMATION ENGINEERING

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Supplementary page: graph template for Question 1 (two copies)

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

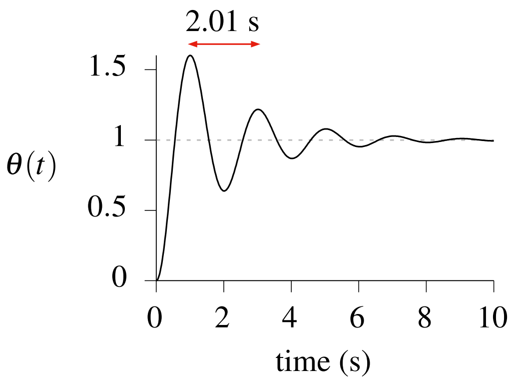
SECTION A

*Answer not more than **two** questions from this section.*

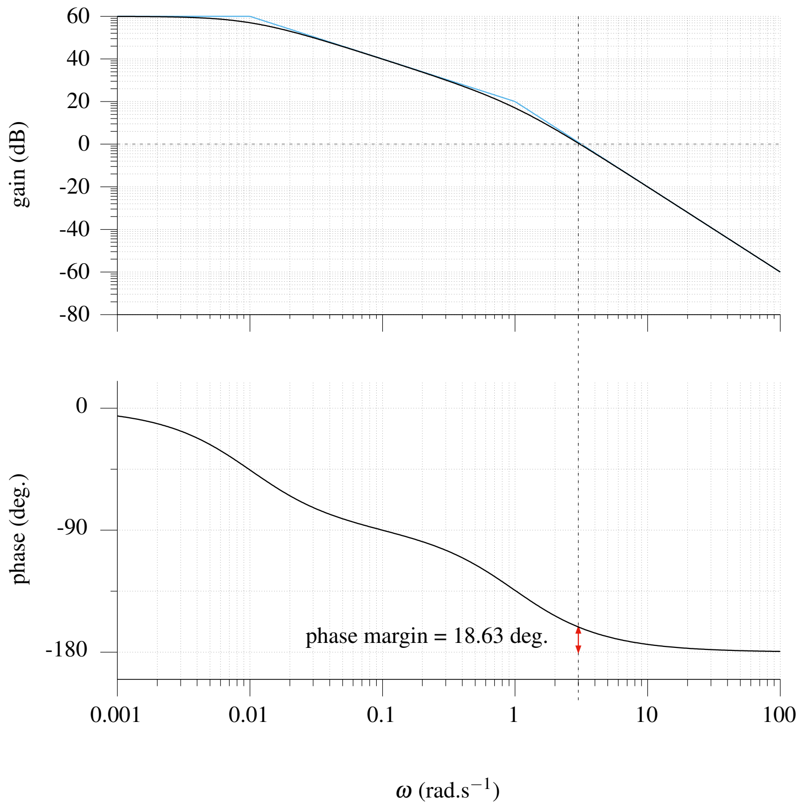
1 (a) The required transfer functions are

- $\bar{\theta}(s) = G_1(s)\bar{\theta}_r(s)$ with $G_1(s) = \dfrac{k_pAB}{\alpha\beta s^2 + (\alpha+\beta)s + k_pAB + 1}$
- $\bar{e}(s) = G_2(s)\bar{\theta}_r(s)$ with $G_2(s) = \dfrac{\alpha\beta s^2 + (\alpha+\beta)s + 1}{\alpha\beta s^2 + (\alpha+\beta)s + k_pAB + 1}$ [6]

(b) (i) The denominator of the closed-loop TF can be identified to $s^2 + 2\zeta\omega_n s + \omega_n^2$ with natural frequency $\omega_n = \sqrt{\frac{k_pAB+1}{\alpha\beta}} \approx 3.16$ rad/s, and damping coefficient $\zeta = \frac{\alpha+\beta}{2\omega_n\alpha\beta} \approx 0.16$. The corresponding step response (e.g. databook) is strongly oscillatory:



(ii) The Bode diagram is shown in the figure below. Asymptotes suffice for determining the phase margin to a reasonable degree of approximation. The phase margin (red arrow) is small, less than 19 degrees, predicting oscillations in the step response.



(c) (i) The steady state error for a step change in reference temperature is equal to $G_2(0) = \frac{1}{1+k_p AB} = \frac{1}{1+1000k_p}$. This must be less than 0.002, giving $k_p \geq \frac{0.998}{20} = 0.499 \approx 0.5$. However, for this value of k_p, the phase margin is barely 25 degrees, much below the 40 degrees required. It is easy to see, by looking at the Bode diagram, that the phase margin is only going to get worse (smaller) as k_p increases. Therefore, these two criteria cannot be simultaneously satisfied by a simple proportional controller.

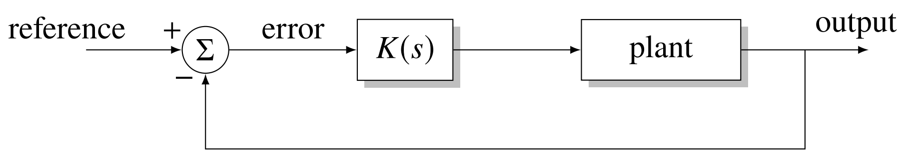
Another (equally valid) answer is to look for the value of k_p that achieves the required phase margin of 40 degrees, and to note that the resulting steady state error is well above the required 0.2%. [4]

(ii) From the Bode diagram, it is easy to see that provided k_d is large enough, (i.e. the cutoff frequency $1/k_d$ is small enough), the phase lag of the open loop will never exceed 90 degrees. Then one does not even have to worry about the changes in gain. For example, any k_d above 1 will work. (A more detailed graphical exploration gives the refined condition $k_d > 0.06$, but this is beyond the answer required from student). [4]

 (TURN OVER

2 (a) $a = G_3$ (e.g. 270 asymptotic phase for large s, characteristic of a 3rd order system); $b = G_4$ (poorly damped 2nd order system); $c = G_5$ (well-damped 2nd order system); $d = G_2$ (semi-circle characteristic of first-order system; to differentiate from similar-looking (e), note that the phase is more negative for the same frequency 1 rad/s, so the cutoff frequency must be smaller); $e = G_1$. [8]

(b) This is a standard negative feedback configuration: [3]



(c) The identity line Re = Im (phase lag of 135 degrees) intersects the Nyquist diagram of the open loop (with $k_p = 1.2$) at a distance roughly $0.6\sqrt{2}$ from the origin. For the phase margin to be 45 degrees (phase lag of 135 degrees), this distance ought to be 1. Therefore, the gain $k_p = 1.2$ should be multiplied by $1/(0.6\sqrt{2})$, which gives a maximum gain of $\sqrt{2}$. [5]

(d) The amplitude of the steady-state response to an output disturbance is given by the modulus of the sensitivity function, $\frac{1}{|1+L(s)|}$. The denominator is the distance between the Nyquist diagram at the relevant frequency ($\omega = 1$ rad/s in this case) and the -1 point on the real axis. Given the information provided in Fig. 3, the best we can do is $|1 + L(0.9j)|^{-1} < |1 + L(j)|^{-1} < |1 + L(1.25j)|^{-1}$; estimating the corresponding distances on the diagram, we conclude that the steady state output amplitude is somewhere between 1.22 and 1.56. [5]

(e) Noting that the outermost circle in Fig. 3 has radius ≈ 1.4, a gain $k_p = 6/7 = 1.2/1.4$ would shrink the Nyquist curve so that this outermost circle becomes the circle of radius 1 – and therefore for $k_p = 6/7$ we have $|L(0.7j)| = 1$. The effect of a time delay τ (Laplace transform $e^{-\tau s}$) is to rotate the Nyquist diagram $\tau\omega$ radians clockwise. To reach instability, the diagram must be rotated ever so slightly more than necessary to intersect the -1 point on the real axis, i.e. enough to encircle this point (cf. Nyquist stability criterion). Looking at the phase lag at $\omega = 0.7$ rad/s, which is roughly -2.18 radians, we conclude that the closed loop becomes unstable when $0.7\tau = (\pi - 2.18) = 0.96$ radians, that is $\tau = 1.37$ s. [5]

3 (a) These are simple, but coupled, first order systems:

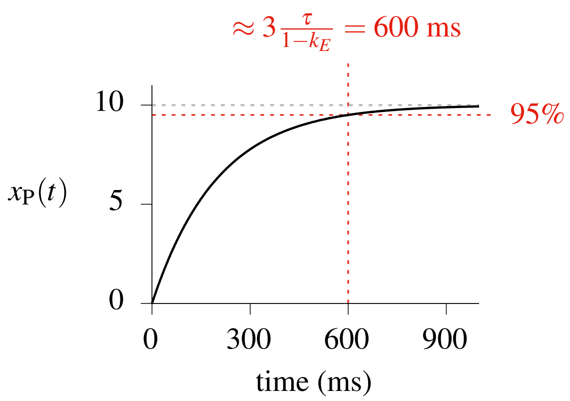
$$\tau \frac{dx_E}{dt} = (\alpha - 1)x_E(t) - \beta x_I(t) + u(t)$$
$$\tau \frac{dx_I}{dt} = \alpha x_E(t) - (1 + \beta)x_I(t)$$

[5]

(b) Here, the inhibitory feedback loop is ignored.

(i) The transfer function of the closed excitatory loop is $G(s) = \frac{1}{(1-\alpha)+\tau s}$. There is a single real pole at $(\alpha - 1)/\tau$; for stability, we require this pole to be negative, i.e. $\alpha < 1$. [3]

(ii) The closed excitatory feedback loop is a first order system, and so the step response is a simple exponential ramp to the steady state value given by $G(0) = 1/(1 - \alpha) = 10$ with time constant equal to the negative inverse of the pole, i.e. $\tau/(1 - \alpha) = 10\tau = 200$ ms:



For an exponential ramp, it takes about three time constants, i.e. 600 ms, to reach 95% of the asymptotic (steady-state) value. [4]

(c) The closed-loop transfer function is now

$$\bar{x}_E(s) = \frac{1 + \alpha + \tau s}{(1 + \tau s)^2}\bar{u}(s)$$

which has a repeated pole $s = -1/\tau$. Therefore, the closed loop is now stable irrespective of the value of α. [5]

(d) The new steady state gain is obtained by evaluating the new TF at $s = 0$; it is equal to $\alpha + 1 = 10$. The timescale on which the closed-loop system reaches its steady-state value in response to an input step is directly related to the pole of the closed-loop TF –

here, this (repeated) pole is $s = -1/\tau$. Thus, it will take approximately 60 ms to reach 95% of the step response. This is 10 times faster than in (a) where the negative feedback loop had been ignored. [4]

(e) The inhibitory feedback loop allows the system to amplify its input without slowing down. In isolation, excitatory feedback can lead to large steady-state gains (here, 10 for $\alpha = 0.9$), but i) there is a need for fine-tuning, as the system approaches instability, and ii) responses inevitably become much slower than the intrinsic time constant τ of the neuron group. In contrast, the full excitatory- inhibitory loop can achieve similarly high steady-state gain of 10 without slowing. [3]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) By using the definition of the inverse Fourier transform of $f(x)$, we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega x}\,d\omega$$
$$f(-\omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x')e^{-jx'\omega'}\,dx'$$
$$2\pi f(-\omega') = \int_{-\infty}^{\infty} F(x')e^{-jx'\omega'}\,dx',$$

where in the second equation above we have replaced x with $-\omega'$ and ω with x'. The last equation shows that the Fourier transform of $F(x)$ is $2\pi f(-\omega)$.

(b) We proceed as

$$G_a(\omega) = \int_{-\infty}^{\infty} g_a(x)e^{-j\omega x}\,dx = \left[\frac{e^{-(j\omega+a)x}}{-j\omega - a}\right]_{x=0}^{x=\infty} - \left[\frac{e^{(a-j\omega)x}}{a - j\omega}\right]_{x=-\infty}^{x=0}$$
$$= \frac{1}{j\omega + a} - \frac{1}{a - j\omega} = -\frac{2j\omega}{a^2+\omega^2}.$$

(c) By using the duality property we have

$$\mathcal{F}\{2/(jx)\} = 2\pi\text{sign}(-\omega).$$

Therefore, $H(\omega) = j\pi\text{sign}(-\omega)$.

(d) (i) By the definition of $Z(\omega)$, we have that

$$Z(\omega - \tau) = \int_{-\infty}^{\infty} z(x)e^{-j(w-\tau)x}\,dx = \int_{-\infty}^{\infty} e^{j\tau x}z(x)e^{-jwx}\,dw.$$

Therefore, $Z(\omega - \tau)$ is the Fourier transform of $e^{j\tau x}z(x)$.

(ii) The convolution of $Z(\omega)$ and $Y(\omega)$ is $\int_{-\infty}^{\infty} Z(\omega - \tau)Y(\tau)\,d\tau$. We now compute the inverse Fourier transform, $n(x)$, of this function:

$$n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} Z(\omega-\tau)e^{jwx}\,d\omega\right]Y(\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} e^{j\tau x}z(x)Y(\tau)\,d\tau$$
$$= 2\pi z(x)y(x).$$

(e) We can use the solution to part (c) (ii) with $z(x) = 1/x$, $Z(\omega) = j\pi\text{sign}(-\omega)$, $y(x) = \text{sinc}(x)$ and $Y(\omega) = \pi\mathbf{I}[-1 \geq \omega \geq 1]$. The required function $M(\omega)$ is then

$$M(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z(\omega - \tau)Y(\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\pi\text{sign}(\tau - \omega)\pi\mathbf{I}[-1 \geq \tau \geq 1]\,d\tau$$

$$= \frac{j\pi}{2}\int_{-1}^{1} \text{sign}(\tau - \omega)\,d\tau = \begin{cases} j\pi & \omega < -1 \\ -j\pi\omega & -1 \leq \omega \leq 1 \\ -j\pi & \omega > 1 \end{cases} .$$

5 (a) The frequency shift theorem says that, given a function $f(t)$ whose Fourier transform is $F(\omega)$, the Fourier transform of $f(t)e^{jn\omega_0 t}$ is equal to $F(\omega - n\omega_0)$. Substituting this into the representation for $x_s(t)$ results in $X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$.

(b) We have that

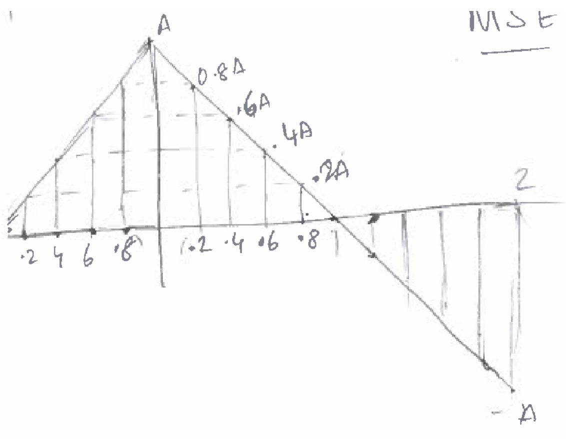
$$X_k = \sum_{n=0}^{N-1} x_n e^{-jkn2\pi/N} \quad \text{for} \quad 0 \geq k \geq N-1\,.$$

and

$$\begin{aligned} X_{N-k} &= \sum_{n=0}^{N-1} x_n e^{-j(N-k)n2\pi/N} = \sum_{n=0}^{N-1} x_n e^{jkn2\pi/N} e^{-jn2\pi} \\ &= \sum_{n=0}^{N-1} x_n e^{jkn2\pi/N} = \sum_{n=0}^{N-1} \left[x_n e^{-jkn2\pi/N}\right]^\star \\ &= \left[\sum_{n=0}^{N-1} x_n e^{-jkn2\pi/N}\right]^\star = [X_k]^\star\,, \end{aligned}$$

for $k = 0, 1, \ldots, N-1$.

(c) (i) The quantisation levels of the uniform quantiser are given by $\{\pm 0.875, \pm 0.625, \pm 0.375, \pm 0.125\}$. Let us denote by $e(0.2k)$ the quantisation error for sample $x(0.2k)$, for $k = 0, 1, 2, \ldots$.



By symmetry, the mean-sq quantisation error is

$$\text{MSE} = (e(0)^2 + 2e(0.8A)^2 + 2e(0.6A)^2 + 2e(0.4A)^2 + 2e(0.8)^2 + e(A)^2)/10\,.$$

With $A = 1$

$$\begin{aligned} \text{MSE} = ((0.125)^2 &+ 2(0.075)^2 + 2(0.025)^2 + \\ &2(0.025)^2 + 2(0.075)^2 + (1 - 0.875)^2)/10 = 5.625 \cdot 10^{-3}\,. \end{aligned}$$

(ii) If $A = 0.1$, $|x(t)| \leq 0.1$, therefore, the positive values of $x(t)$ are quantised to the level $+0.125$, and the negative values are all quantised to -0.125. The MSE is

$$\text{MSE} = (0.125)^2 + 2(0.08 - 0.125)^2 + 2(0.06 - 0.125)^2 + 2(0.04 - 0.125)^2 + 2(0.02 - 0.125)^2 + (0.1 - 0.125)^2)/10 = 0.006525\,.$$

(iii) Signal power: $\int_0^1 A^2 t^2\, dt = A^2/3$. SNR for $A = 1$: $10\log_{10}(1/3/(5.625 \cdot 10^{-3})) = 17.7\,\text{dB}$ and for $A = 0.1$: $10\log_{10}(0.01/3 \times 1/0.006525) = -2.92\,\text{dB}$, which is bad. To obtain good SNR at both high and low amplitues, one can use non-uniformly spaced quantisation levels. For example, a δ-level quantiser with levels $\{\pm 0.7, \pm 0.4, \pm 0.2, \pm 0.05\}$.

6 (a) Rate $= 2\ \text{bits}/T\ \text{sec} = 2 \times 10^5 \text{bits}/s$

(b)

$$\int p(t)^2\, dt = \frac{3}{T} \times 2 \int_0^{T/2} \left(1 - \frac{t}{T/2}\right)^2 dt = \frac{6}{T} \times \frac{T}{2} \int_0^1 u^2 du = 1\,.$$

Therefore, $p(t)$ has unit energy.

(c) The received signal $y(t) = x(t) + u(t)$,

$$y(t) = \sum_k x_k p(t - kT) + n(t)\,, \quad \text{(n(t) is the noise waveform)}\,.$$

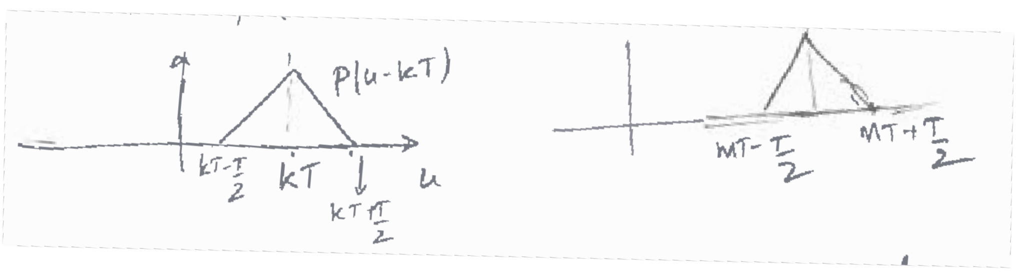
The filter output is

$$\gamma(t) = y(t) * p(-t) = \sum_k x_k (p(t - kT) * p(-t)) + n(t) * p(-t)$$
$$= \sum_k x_k \int_{-\infty}^{\infty} p(u - kT)p(u - t)\, du + \int n(u)p(u - t)du\,.$$

If there is no noise

$$\gamma(mT) = \sum_k x_k \int p(u - kT)p(u - mT)\, du\,.$$

Now, when $k \neq m$, $p(u - kT)$ and $p(u - mT)$ have no overlap:

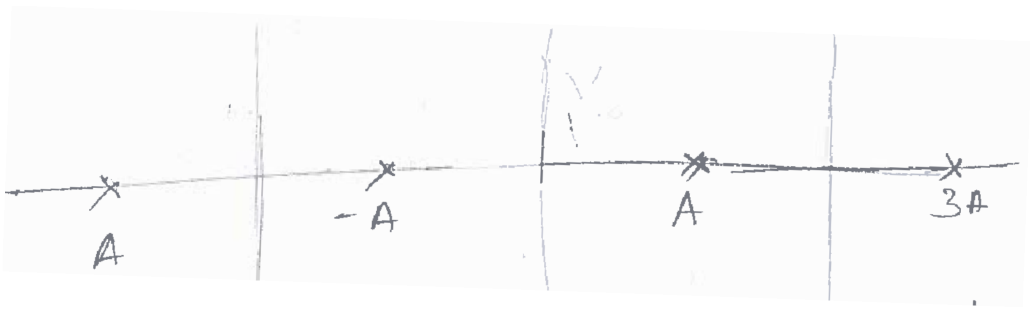


When $k = m$,

$$\int p(u - mT)^2\, du = \int p(u)^2\, du = 1 \Rightarrow \gamma(mT) = x_n + \sum_{k \neq m} x_k \times 0\,.$$

(d) The optimum decision rule is

$$\hat{y}_m = \begin{cases} 3A & \text{if } y_m \geq 2A \\ A & \text{if } \Theta \leq y_m < 2A \\ -A & \text{if } -2A < y_m < \Theta \\ -3A & \text{if } y_m \leq -2A \end{cases} .$$



(e) $p(\hat{x} \neq x|x = 3A) = p(3A + N < 2A|x = 3A) = p(N < -A|x = 3A) = p(N < -A) = Q(A/\sigma)$. $p(\hat{x} \neq x|x = A) = p(\{A + N > 2A\} \cup \{A + N < 0\}) = p(\{N > A\} \cup \{N < -A\}) = 2Q(A/\sigma)$. By symmetry, $p(\hat{x} \neq x|x = -3A) = p(\hat{x} \neq x|x = 3A)$ and $p(\hat{x} \neq x|x = A) = p(\hat{x} \neq x|x = -A)$. Thereore, the overall proability of detection error is $p_e = 1/4(Q(A/\sigma) + 2Q(A/\sigma) + 2Q(A/\sigma) + Q(A/\sigma)) = 3/2Q(A/\sigma)$. The average energy per symbol E_s is

$$E_s = \frac{(3A)^2 + A^2 + (-A)^2 + (-3A)^2}{4} = \frac{20A^2}{4} = 5A^2 . \quad (1)$$

$E_s = E_b \log_2 4$ (each symbol carries 2 bits) $\Rightarrow E_b = 5A^2/2 \Rightarrow A = \sqrt{2E_b/5}$. Therefore, $p_e = 3/2Q(\sqrt{2E_b/(5\sigma^2)})$.

(f) $p(t)$ is the convolution of a rectangular function with itself: If $q(t)$ takes value $\sqrt{2/T}$ for $0 \geq t \geq T/2$ and zero otherwise, then $p(t) = \sqrt{3/T}q(t) * q(t)$. Therefore, $p(f) = \sqrt{3/T}(Q(f))^2 \Rightarrow |p(f)|^2 = 3/T|Q(f)|^4$. Then, $|Q(f)| = \sqrt{T/2}|\text{sinc}(\pi f T/2)| \Rightarrow |p(f)|^2 = 3T/4\text{sinc}^4(\pi f T/2)$ and $S_x(f) = E_s/T|p(f)|^2 = 15A^2/4\text{sinc}^4(\pi f T/2) \approx 1/f^4$

END OF PAPER

Numerical Answers

1b(ii) Small phase margin just under 19 degrees.

1c(ii) Any $k_d > 0.06$ works.

2a $a = G_3, b = G_4, c = G_5, d = G_2, e = G_1$.

2c Maximum k_p is $\sqrt{2}$.

2d Lower bound: 1.22; upper bound: 1.56.

2e Maximum delay $\tau = 1.37$s.

3b(i) Approximately 600 ms.

3d(i) Approximately 60 ms.

5c(i) MSE = $5.625 \cdot 10^{-3}$.

5c(ii) MSE = $6.525 \cdot 10^{-3}$.

5c(iii) Signal power: $A^2/3$. SNR for $A = 1$: 17.7 dB. SNR for $A = 0.1$: −2.92 dB.

6a $2 \cdot 10^5$ bits/s.

6e Average energy per symbol: $E_s = 5A^2$, then $p_e = 3/2Q(\sqrt{2E_b/(5\sigma^2)})$.