EGT1
ENGINEERING TRIPOS PART IB

Thursday 10 June 2021   11:00 to 13:10

Paper 6

INFORMATION ENGINEERING

This is an ***open-book*** *exam.*

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each subsequent answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
You have access to the Engineering Data Book, online or as your hard copy.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) (i) $a = G_3, b = G_1, c = G_5, d = G_2, e = G_4.$ [4]

(ii) $f = G_1, g = G_3, h = G_4, i = G_5, k = G_2.$ [4]

(b) In the Laplace domain, $\bar{x}(s) = G_4(s)\bar{u}(s)$ implies $(100s^2 + 101s + 1)\bar{x}(s) = \eta s\bar{u}(s)$. Using basic properties of the Laplace transform, e.g. that $s\bar{z}(s)$ is the Laplace transform of $\frac{dz}{dt}$, we can express this relationship in the time domain as the following second-order linear ordinary differential equation:

$$100\frac{d^2x}{dt^2} + 101\frac{dx}{dt} + x = \eta\frac{du}{dt}.$$

[4]

(c) (i) The full Bode diagram of the return ratio is shown in Fig. 2. [5]

(ii) To compute the phase margin, we identify the angular frequency at which the gain of the return ratio is 1 (0 dB), and look up the corresponding phase. The phase margin is the difference from this phase to -180 degrees. Here, the phase margin is slightly above 11 degrees – see red annotations in Fig. 2. [4]

(iii) To find the value of k_p that achieves a phase margin of 45 degrees, we identify the frequency at which the phase is $-180 + 45 = -135$ degrees. For this phase, the gain of the return ratio with $k_\mathrm{p} = 1$ is roughly 14 (23 dB) – see blue annotations in Fig. 2. Therefore, we would need a k_p of $1/14 \approx 0.07$ for the phase margin to be 45 degrees. [4]

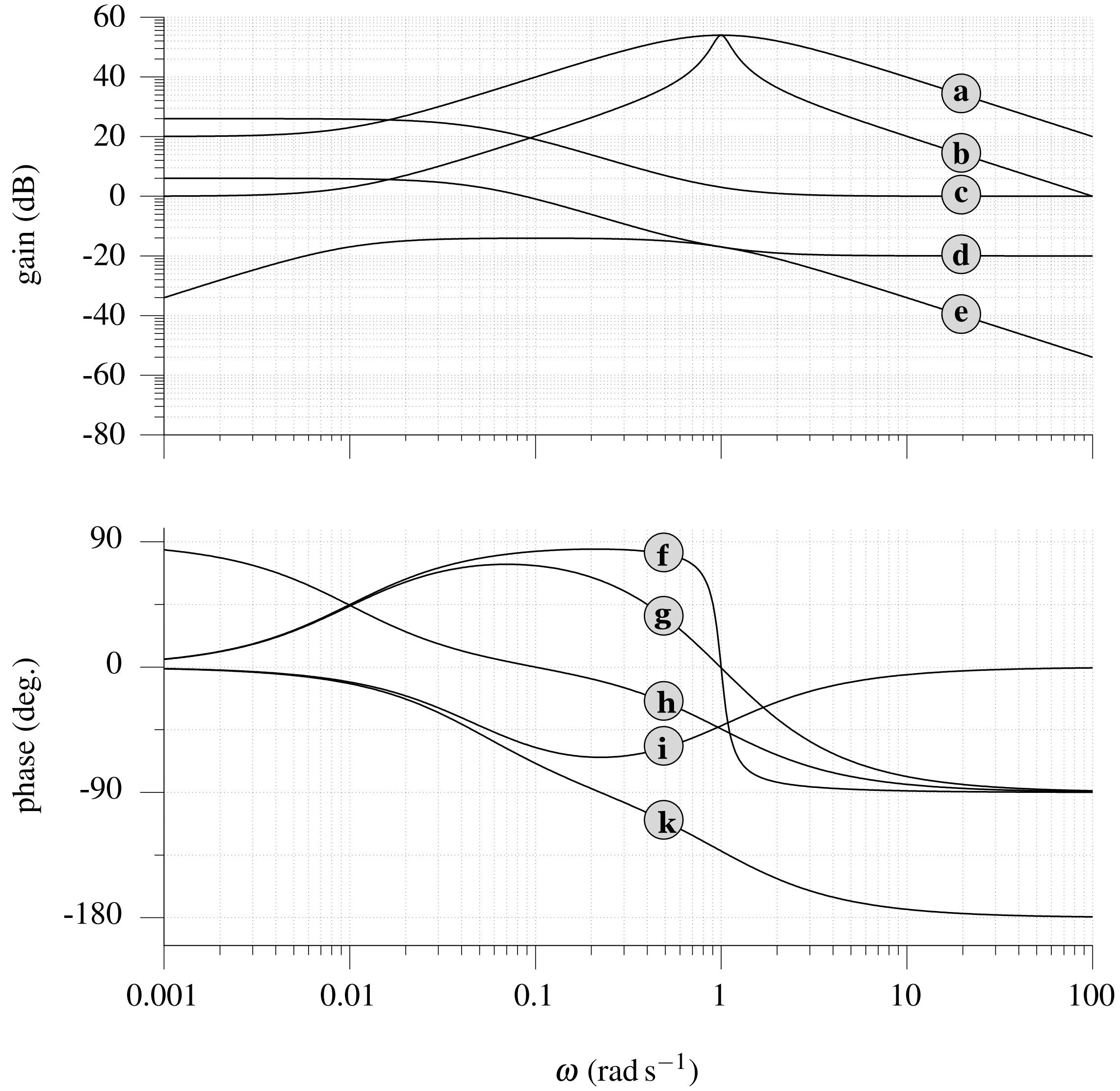

Fig. 1

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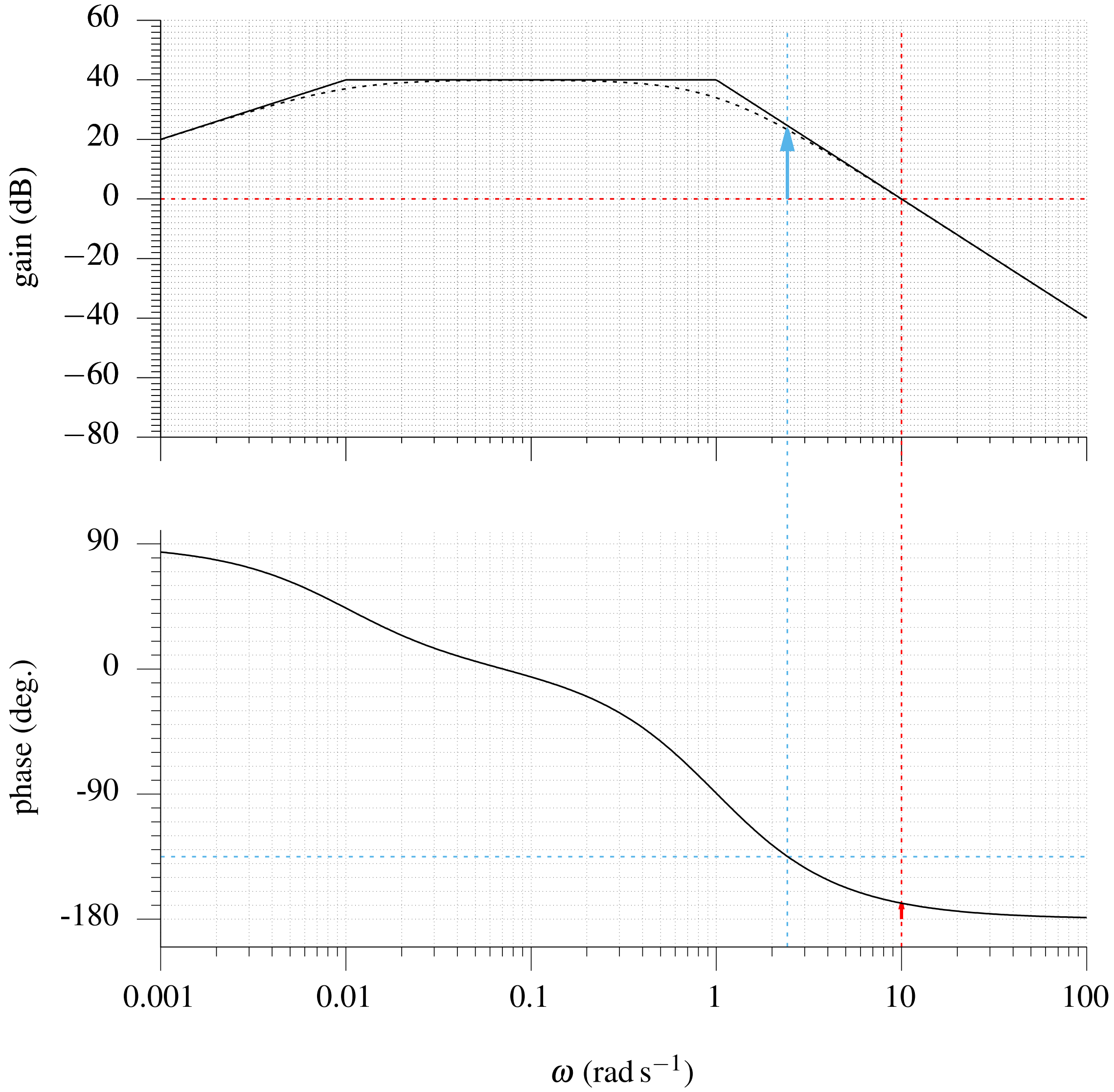

60
40
20
0
−20
−40
−60
−80
gain (dB)
90
0
-90
-180
phase (deg.)
0.001
0.01
0.1
1
10
100
ω (rad s⁻¹)

Fig. 2

(cont.

2 (a) Since the diagram approaches the origin from the left at large frequencies (i.e. with a phase approaching -180 degrees), $G(s)$ has at least two poles (each contributing a -90-degree phase loss). If there was more poles, they would need to be compensated by zeros (otherwise the phase would be even lower at high frequencies), but since we are told $G(s)$ has no zeros, we conclude it has only two poles. [3]

(b) In closed loop, we have $\bar{e}(s) = \dfrac{1}{1 + K(s)G(s)}\bar{x}(s)$. [3]

(c) From the diagram, $G(0) = 1$. Therefore the steady state error is $(1 + k_{\mathrm{p}}G(0))^{-1} = (1 + k_{\mathrm{p}})^{-1}$. For this to be 0.02, we need $k_{\mathrm{p}} = 49$ (50 would be an acceptable answer). [5]

(d) To obtain the Nyquist diagram of the return ratio for $k_{\mathrm{p}} = 49$, we would need to scale that of $30G(s)$ by $49/30 \approx 1.63$. Thus, in order to find the requested phase margin, we need to look for the intersection of the Nyquist diagram of $30G(s)$ with the circle of radius $30/49 \approx 0.61$. The phase margin is the angle made by this point with the (negative)horizontal, and is about 16 degrees here. [5]

(e) (i) The steady state error in step response is zero, because of the presence of an integrator in the loop (the $1/s$ term in the controller). [4]

(ii) For small ω, the return ratio is approximately

$$\begin{aligned} K(j\omega)G(j\omega) &\approx \frac{(1 - 20j\omega)(j\omega + \alpha)}{j\omega} \\ &\approx \frac{-1}{\omega}(j + 20\omega)(j\omega + \alpha) \\ &\approx \frac{-1}{\omega}(20\alpha\omega - \omega + j(\alpha + 20\omega^2)) \\ &\approx -(20\alpha - 1) + j[\ldots] \end{aligned}$$

We are told that the Nyquist diagram of $K(s)G(s)$ asymptotes onto the y-axis for small ω, meaning that the the real part must be zero. Thus, $\alpha = 1/20$. [5]

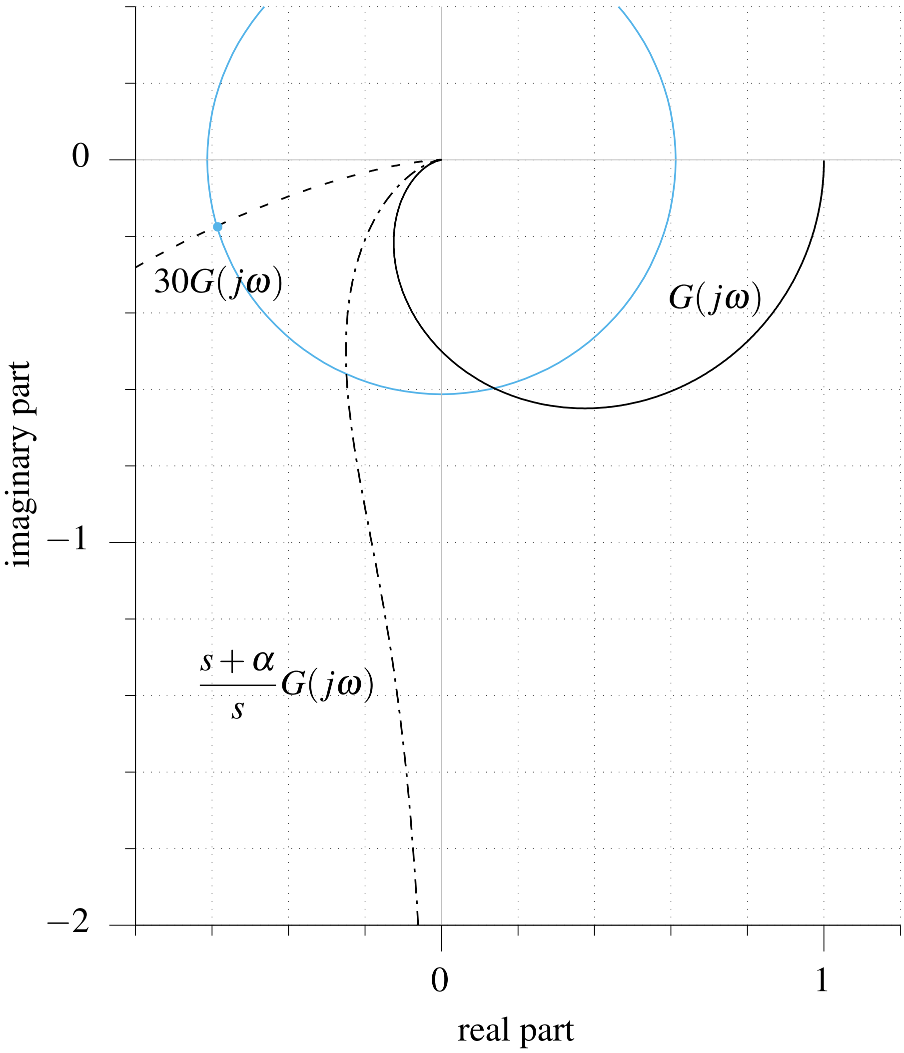

0
−1
−2
imaginary part
$30G(j\omega)$
$G(j\omega)$
$\frac{s+\alpha}{s}G(j\omega)$
0
1
real part

Fig. 3

3 (a) Applying the Laplace transform on both sides of Eq.1, we obtain

$$(Is^2 + cs + k)\bar{\theta}(s) - k\bar{u}(s) = \bar{\tau}(s)$$

For the transfer function from u we take $\tau = 0$, thus

$$\bar{\theta}(s) = G(s) = \frac{k}{s^2 + cs + k}\bar{u}(s)\ .$$

For stability, poles must have negative real part. From the transfer function, poles are $\frac{1}{2}\left(-c \pm \sqrt{c^2 - 4k}\right)$. Take $c > 0$ and consider the case $c^2 - 4k \geq 0$. Since $\sqrt{c^2 - 4k} < \sqrt{c^2}$ for $k > 0$, it follows that $-c \pm \sqrt{c^2 - 4k} < 0$, which guarantees that poles have negative real part. Consider now the case $c^2 - 4k < 0$. This gives complex conjugated poles whose real part is $-c$. [3]

(b) Given the open-loop transfer function $\bar{\theta}(s) = G(s) = \dfrac{4}{s^2 + 4s + 4}\bar{u}(s)$ and the input $\bar{u}(s) = \frac{1}{s}$, the step response reads

$$\bar{\theta}(s) = \frac{4}{s^3 + 4s^2 + 4s} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

where $A = [s\bar{\theta}(s)]_{s=0} = G(0) = 1$, $B = [(s+2)^2\bar{\theta}(s)]_{s=-2} = \frac{4}{-2} = -2$, and $A + C = \lim_{s\to\infty} s\bar{\theta}(s) = \lim_{s\to\infty} \frac{4}{s^2+4s+4} = 0$, thus $C = -1$. By Laplace anti-transform

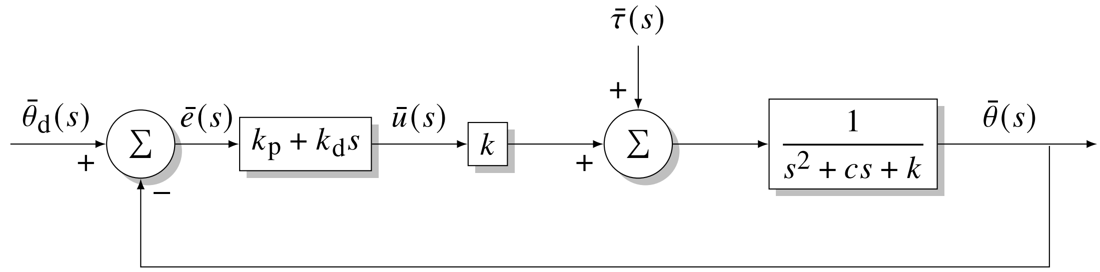
$$\theta(t) = 1 - 2te^{-2t} - e^{-2t} \qquad \text{for } t \geq 0\ .$$

[4]

(c) Note that $G(s) = \dfrac{4}{s^2 + 4s + 4} = \dfrac{1}{\frac{s^2}{4} + s + 1} = \dfrac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1}$ for $\omega_n = 2$ and $\zeta = 1$.

So, the step response can be read in Section 4.4. of the Mechanics handbook.
For generic k, $\omega_n = \sqrt{k}$ and $\frac{2\zeta}{\sqrt{k}} = \frac{4}{k}$, that is, $\zeta = \frac{1}{2\sqrt{k}}$. Thus, $k > 4$ makes the response less damped / more oscillatory. The frequency of oscillations also increases. Likewise, $k < 4$ makes the response more damped / slower. [4]

(d) (i) Below is a block diagram of the closed loop:



[2]

(ii) Using the return ratio $L(s) = k\dfrac{k_\mathrm{p} + k_\mathrm{d}s}{s^2 + cs + k}$, the sensitivity function[1] reads

$$S(s) = \frac{1}{1 + L(s)} = \frac{s^2 + cs + k}{s^2 + (c + kk_\mathrm{d})s + k(1 + k_\mathrm{p})} = \frac{s^2 + 4s + 4}{s^2 + 4(1 + k_\mathrm{d})s + 404}\ .$$

Therefore, the denominator of any closed loop transfer function is

$$s^2 + 4(1 + k_\mathrm{d})s + 404 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

and we want to find k_d to satisfy $\zeta = 1$. Thus, $\omega_n = \sqrt{404} \simeq 20.1$ and $2\zeta\omega_n = 4(1 + k_\mathrm{d})$. From the latter, setting the damping ratio $\zeta = 1$, one gets $2\sqrt{404} = 4(1 + k_\mathrm{d})$, that is, $k_\mathrm{d} = \frac{\sqrt{404}}{2} - 1 \simeq 9.05$. [4]

(iii) From the block diagram, the transfer function from $\bar{\tau}(s)$ to $\bar{\theta}(s)$ reads

$$T_{\tau\to\theta}(s) = \frac{\frac{1}{s^2+4s+4}}{1 + \frac{4(k_\mathrm{p}+k_\mathrm{d}s)}{s^2+4s+4}} = \frac{1}{s^2 + 4(1 + k_\mathrm{d})s + 4(1 + k_\mathrm{p})} = \frac{1}{s^2 + 40.2s + 404}.$$

Thus, $T_{\tau\to\theta}(0) = \frac{1}{404} \leq 0.0025 < 0.01$, as required. [4]

(e) For $\tau = -\gamma\dot{\theta} + \tau_\mathrm{ext}$, the denominator of any closed loop transfer function becomes

$$s^2 + (4 + \gamma + 4k_\mathrm{d})s + 404$$

so the damping ratio increases for $\gamma > 0$, which induces slower transients. The steady state to step inputs does not change. [4]

[1]Any closed-loop transfer function could be used.

SECTION B

Answer not more than ***two*** *questions from this section.*

4 (a) From the definition of the inverse Fourier transform We know that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{j\omega t}\, d\omega\,.$$

Substituting αt for t, we obtain

$$f(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2} e^{j\omega\alpha t}\, d\omega\,.$$

We then substitute $\omega' = \alpha\omega$ and $d\omega = (1/\alpha)d\omega'$, resulting in

$$f(\alpha t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|\alpha|}\frac{2a}{a^2+(\omega')^2/\alpha^2} e^{j\omega' t}\, d\omega'\,.$$

Therefore, the Fourier transform of $f(\alpha t)$ is then $2a/(|\alpha|a^2+\omega^2/|\alpha|)$, where the abosolute value in α has been introduced to account for the fact that the integration limits get swapped around when $\alpha < 0$. [4]

(b) The convolution theorem says that if $h(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$, then the Fourier transform $H(\omega)$ of $h(t)$ satisfies $H(\omega) = F(\omega)G(\omega)$ where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$, respectively. Therefore, we have that $Z(\omega) = Y(\omega)^2$, where $Y(\omega)$ is the Fourier transform of the rectangular pulse $y(t)$. In particular, $Y(\omega) = \mathrm{sinc}(\omega/2)$, so that $Z(\omega) = \mathrm{sinc}^2(\omega/2)$. [4]

(c) The function $x(t)$ can be written as $x(t) = g(t) + h(t)$ where

$$g(t) = \begin{cases} 0.5 & -T \le x \le T \\ 0 & \text{otherwise} \end{cases}, \qquad h(t) = \begin{cases} x0.5/T + 0.5 & -T \le x < 0 \\ 0.5 - x0.5/T & 0 \le x \le T \\ 0 & \text{otherwise} \end{cases}.$$

The function $g(t)$ is a rectangular pulse centered at the origin and of width $2T$ and height 0.5. The Fourier transform $G(\omega)$ of $g(t)$ is then $T\mathrm{sinc}(\omega T)$. The function $h(t)$ is a triangular pulse centered at the origin and of height 0.5 and width $2T$. That is, $h(t) = 0.5z(t/T)$, where $z(t)$ is the function from part (b). Using the result from part (a), we have that the Fourier transform of $h(t)$ is then $H(\omega) = 0.5TZ(\omega T) = 0.5T\mathrm{sinc}^2(\omega T/2)$. The Fourier transform of $x(t)$ is then given by $X(\omega) = H(\omega)+G(\omega) = 0.5T\mathrm{sinc}^2(\omega T/2)+T\mathrm{sinc}(\omega T)$. [4]

(d) (i) By linearity, the Fourier transform of $e^{-2|t|}/4$ is $1/(4+\omega^2)$. By duality, the Fourier transform of $g(t) = 1/(4+t^2)$ is $G(\omega) = \pi e^{-2|\omega|}/2$. [4]

(ii) We have that $\sin(2t) = (e^{2t} - e^{-2t})/(2i)$ and, by using the frequency shift theorem, we have that the Fourier transform of $m(t) = \sin(2t)/(4+t^2)$ is $M(\omega) = (G(\omega-2) - G(\omega+2))/(2i) = i\pi e^{-2|\omega+2|}/4 - i\pi e^{-2|\omega-2|}/4$. [3]

(e) Using Parseval's theorem, we have that

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 \, d\omega \, ,$$

where $S(\omega)$ is the Fourier transform of $s(t)$. We have that $s(t)$ is the product of $\cos(t)$ and $\mathrm{sinc}(t)$, whose Fourier transforms are, respectively, $\pi[\delta(\omega+1) + \delta(\omega-1)]$ and $\pi\mathbf{I}[-1 \le \omega \le 1]$, where $\mathbf{I}[x]$ is an indicator function that takes value 1 if x is true and 0 otherwise. Therefore, we have that

$$\begin{aligned} S(\omega) &= \frac{\pi}{2} \int_{-\infty}^{\infty} \mathbf{I}[-1 \le \omega - \tau \le 1] \, \{\delta(\tau+1) + \delta(\tau-1)\} \, d\tau \\ &= \frac{\pi}{2} \{\mathbf{I}[-1 \le \omega - 1 \le 1] + \mathbf{I}[-1 \le \omega + 1 \le 1]\} \\ &= \frac{\pi}{2} \{\mathbf{I}[0 \le \omega \le 2] + \mathbf{I}[-2 \le \omega \le 0]\} \, . \end{aligned}$$

The energy of $s(t)$ is then

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\frac{\pi}{2} \{\mathbf{I}[0 \le \omega \le 2] + \mathbf{I}[-2 \le \omega \le 0]\} |^2 \, d\omega = \frac{\pi}{2} \, .$$

[6]

5 (a) (i) The sampled signal is given by

$$x_p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(t) e^{jn2\pi t/T} .$$

The Fourier transform of the sampled signal, denoted by $X_p(\omega)$, can then be calculated by using the frequency shift theorem and by using additivity. In particular,

$$X_p(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n2\pi/T) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-2|\omega - n2\pi/T|} .$$

We will not be able to reconstruct the original signal from the sampled one because there is overlap between the periodic repetitions of $X(\omega)$ in the spectrum of the sampled signal. [3]

(ii) The energy of the original signal is

$$E = \frac{2}{2\pi} \int_0^{\infty} X(\omega)^2 \, d\omega = \frac{1}{2\pi} (-2e^{-4\omega}/4)|_0^{\infty} = \frac{1}{4\pi} .$$

If the lowpass filter has cutoff frequency ω_{max}, the energy of the filtered signal is

$$E_{\text{filtered}} = \frac{2}{2\pi} \int_0^{\omega_{max}} X(\omega)^2 \, d\omega = \frac{1}{2\pi} (-2e^{-4\omega}/4)|_0^{\omega_{max}} = \frac{1}{4\pi} \left(1 - e^{-4\omega_{max}}\right) .$$

The value of ω_{max} that makes the filtered signal keep 80% of the energy is, therefore,

$$0.4 = \frac{1}{2} \left(1 - e^{-4\omega_{max}}\right) \rightarrow \omega_{max} = -\log(0.02)/4 \approx 0.402 .$$

[4]

(iii) The spectrum of the filtered sampled signal is equal to the spectrum of the original filtered signal added many times to itself, but each time displaced by an amount equal to a multiple of $2\pi/T$. The maximum frequency content of the sampled signal is $\omega_{max} = 0.402$. Overlap between the frequency spectrums will not occur when $2\pi/T = 2\omega_{max}$. Therefore, we should choose $T = \pi/0.402 \approx 7.81$. [3]

(b) (i) From parts (a)(i) and (a)(ii) we know that the Fourier transform of the sampled signal repeats after $\omega > 2\pi/T$. The discrete Fourier transform evaluates the spectrum at N evenly spaced values in the interval $[0, (N-1)2\pi/(NT)]$. These are given by $\omega_i = (i-1)2\pi/(TN)$ for $i = 0, \ldots, N-1$. When $T = 50$ seconds and $N = 4$, we obtain $\omega_0 = 0$ rad/s, $\omega_1 = 0.031$ rad/s, $\omega_2 = 0.063$ rad/s and $\omega_3 = 0.094$ rad/s. [3]

(ii) The inverse DFT is given by

$$x_n = \frac{1}{4} \sum_{m=0}^{3} X_m e^{j2\pi nm/4} = \frac{1}{2} j e^{j2\pi n/4} - \frac{1}{2} j e^{j2\pi n3/4} .$$

We, therefore, obtain

$$\begin{aligned}
x_0 &= \frac{1}{2}je^{j2\pi 0/4} - \frac{1}{2}je^{j2\pi 0/4} = 0\,,\\
x_1 &= \frac{1}{2}je^{j2\pi 1/4} - \frac{1}{2}je^{j2\pi 3/4} = -0.5 - 0.5 = -1\,,\\
x_2 &= \frac{1}{2}je^{j2\pi 2/4} - \frac{1}{2}je^{j2\pi 6/4} = -0.5j + 0.5j = 0\,,\\
x_3 &= \frac{1}{2}je^{j2\pi 3/4} - \frac{1}{2}je^{j2\pi 9/4} = 0.5 + 0.5 = 1\,.
\end{aligned}$$

[5]

(c) The power of the signal $x(t)$ is given by

$$\text{signal power} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} (0.5x)^2\,dx = 0.5^2\frac{1}{2}\int_0^2 x^2\,dx = 0.5^2\frac{1}{2}2^3/3 = 1/3\,. \quad (1)$$

The measured signal values at $t = \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2\}$ are $x = \{0.000, 0.125, 0.250, 0.375, 0.500, 0.625, 0.750, 0.875, 1\}$. The quantised levels are $\{-0.833, -0.5, -0.166, 0.166, 0.5, 0.833\}$. The average MSE of the quantised signal is then

$$\begin{aligned}
\text{MSE} = {}&((0.166)^2 + 2(0.125 - 0.166)^2 + 2(0.250 - 0.166)^2 + 2(0.375 - 0.5)^2 +\\
&2(0.5 - 0.5)^2 + 2(0.625 - 0.5)^2 + 2(0.750 - 0.833)^2 +\\
&2(0.875 - 0.833)^2 + (1 - 0.833)^2)/16\\
={}& 0.0095\,.
\end{aligned}$$

The SNR is then

$$\text{SNR} = \frac{0.333}{0.0164} = 15.431\ \text{dB}\,.$$

[7]

6 (a) (i) The modulation index is $\max_t |s(t)|$ divided by 15. Since the modulation index has to be 0.5 and $f(t)$ is at most ± 1, we obtain $a = 0.5 \times 15 = 7.5$. The radio channel is centered at 900kHz. Therefore, we have that $\omega_c = 900,000 \times 2\pi = 5,654,867$ rad/s. [4]

(ii) The Fourier transform of $\cos(\omega_c t)$ is $\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$. The Fourier transform of the product is the Fourier transform of the convolution divided by 2π. This results in

$$S(\omega) = 15\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] + \frac{a}{2}[F(\omega - \omega_c) + F(\omega + \omega_c)] .$$

[4]

(iii) We need a bandwidth of $2 \times$ 5kHz + $\times$3kHz = 16kHz (13kHz if students assume that the bandwidth gap is shared between consecutive channels). [2]

(iv) $s(t)$ is amplitude modulation (AM) and $u(t)$ is double side-band suppressed carrier (DSB-SC). $u(t)$ has smaller power (as we do not transmit the carrier) but needs more complex circuitry at the receiver to demodulate. $s(t)$ can be demodulated using a simple envelope detector. [4]

(b) The transmission rate in bits per second is $1/10^{-4} = 1000$. [2]

(c) (i) Note that $p(X_k = A) = 2/3$, $p(X_k = A) = 1/3$ and $p(Y_k|X_k) = \mathcal{N}(X_k, |\sigma^2)$. The decision threshold θ satisfies

$$p(X_k = A)p(Y_k = \theta|X_k = A) = p(X_k = -A)p(Y_k = \theta|X_k = -A)$$
$$2/3 \times \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{(\theta - A)^2}{2\sigma^2}\right\} = 1/3 \times \frac{1}{\sqrt{\sigma^2}} \exp\left\{-\frac{(\theta + A)^2}{2\sigma^2}\right\} ,$$

which results in

$$\theta = -0.5\sigma^2 \log(2)/A .$$

[5]

(ii) The probability of error is the probability of receiving A and decoding $-A$ plus the probability of receiving $-A$ and decoding A. That is,

$$\begin{aligned} p(\text{Error}) &= p(X_k = A)p(Y_K < \theta) + p(X_k = -A)p(Y_K > \theta) \\ &= 2/3p(A + N_k < \theta) + 1/3p(-A + N_k > \theta) \\ &= 2/3p(A/\sigma + N_k/\sigma < \theta/\sigma) + 1/3p(-A/\sigma + N_k/\sigma > \theta/\sigma) \\ &= 2/3p(N_k/\sigma < (\theta - A)/\sigma) + 1/3p(N_k/\sigma > (\theta + A)/\sigma) \\ &= 2/3[1 - Q((\theta - A)/\sigma)] + 1/3Q((\theta + A)/\sigma) . \end{aligned}$$

[4]

END OF PAPER